GLOBAL HÖLDER CONTINUITY OF SOLUTIONS TO QUASILINEAR EQUATIONS WITH MORREY DATA

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Abstract. We deal with general quasilinear divergence-form coercive operators whose prototype is the m-Laplacean operator. The nonlinear terms are given by Carathéodory functions and satisfy controlled growth structure conditions with data belonging to suitable Morrey spaces. The fairly non-regular boundary of the underlying domain is supposed to satisfy a capacity density condition which allows domains with exterior corkscrew property.

We prove global boundedness and Hölder continuity up to the boundary for the weak solutions of such equations, generalizing this way the classical L^p-result of Ladyzhenskaya and Ural'tseva to the settings of the Morrey spaces.

1. Introduction

The general aim of the present article is to get sufficient conditions ensuring boundedness and Hölder continuity up to the boundary for the weak solutions to general quasilinear equations with discontinuous ingredients which are controlled within the Morrey functional scales. Precisely, we deal with weak solutions $u \in W_0^{1,m}(\Omega)$ of the Dirichlet problem

$$\begin{cases} \operatorname{div}\big(\mathbf{a}(x,u,Du)\big) = b(x,u,Du) & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with generally non-smooth boundary, $m \in (1,n]$, and $\mathbf{a}\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and $b\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ are Carathéodory maps. Let us stress the reader attention at the very beginning that prototypes of the quasilinear equations studied are these for the *m-Laplace* operator $\operatorname{div}\big(|Du|^{m-2}Du\big)$ with $m>1$, or these for the *m-area* type operator $\operatorname{div}\Big(\big(A+|Du|^2\big)^{\frac{m-2}{2}}Du\Big)$ with $m \geq 2$ and $A>0$.

Regarding the nonlinear terms in (1.1), we assume *controlled growths* with respect to u and Du, that is,

$$\begin{cases} |\mathbf{a}(x,u,Du)| = \mathcal{O}\left(\varphi(x) + |u|^{\frac{\ell(m-1)}{m}} + |Du|^{m-1}\right),\\ |b(x,u,Du)| = \mathcal{O}\left(\psi(x) + |u|^{\ell-1} + |Du|^{\frac{m(\ell-1)}{\ell}}\right) \end{cases}$$

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as $|z|, |Du| \to \infty$, where ℓ is the Sobolev conjugate of m, and *coercivity* of the differential operator considered

$$\mathbf{a}(x,u,Du)\cdot Du \geq \gamma|Du|^m - \Lambda|u|^\ell - \Lambda\varphi(x)^{\frac{m}{m-1}}$$

with non-negative functions φ and ψ and constants $\gamma > 0$ and $\Lambda \geq 0$. It is worth noting that $\varphi \in L^{\frac{m}{m-1}}(\Omega)$ and $\psi \in L^{\frac{nm}{nm+m-n}}(\Omega)$, together with the controlled growths, are the *minimal* hypotheses on the data under which the concept of $W_0^{1,m}(\Omega)$-weak solution to (1.1) makes sense. In what follows, we will assume that φ and ψ are non-negative measurable functions belonging to suitable *Morrey spaces*. Namely, we suppose

$$\begin{aligned} &\varphi \in L^{p,\lambda}(\Omega) \quad \text{with } p > \tfrac{m}{m-1}, \qquad \lambda \in (0,n) \text{ and } (m-1)p + \lambda > n \\ &\psi \in L^{q,\mu}(\Omega) \quad \text{with } q > \tfrac{mn}{mn+m-n}, \quad \mu \in (0,n) \text{ and } mq + \mu > n. \end{aligned} \tag{1.2}$$

The non-regular boundary of Ω will be assumed to satisfy a *density condition* expressed in terms of variational P-capacity for some $P \in (1,m)$ (see (2.1) below), which requires the complement $\mathbb{R}^n \setminus \Omega$ to be *uniformly P-thick*. This notion is a natural generalization of the *measure density condition,* known also as *(A)-condition* of Ladyzhenskaya and Uraltseva (cf. [12, 13, 14]), which holds for instance when each point of $\partial\Omega$ supports the *exterior cone property*, excluding this way exterior spikes on $\partial\Omega$. In that sense, the uniform P-thickness condition is satisfied by domains with C^1-smooth or Lipschitz continuous boundaries, but it holds also when $\partial\Omega$ is flat in the sense of Reifenberg, including this way boundaries with fractal structure such as the von Koch snowflake. Anyway, the class of domains verifying the capacity density condition (2.1) goes beyond these common examples and contains for example sets with boundaries which support the *uniform corkscrew* condition.

The regularity problem for solutions to (1.1) has been a long-standing problem in the PDEs theory, related to the Hilbert 19th Problem. In particular, the task to get Hölder continuity of the weak solutions under very general hypotheses on the data is a first step towards developing relevant solvability and regularity theory for (1.1) in the framework of various functional scales (see for instance [3, 4, 21] and the references therein). In case when (1.1) is the Euler–Lagrange equation of a given functional $\mathcal{F}$ that is the problem of regularity of the minimizers of $\mathcal{F}$ and this links (1.1) to important equations from differential geometry or mathematical physics, such as Gunzburg–Landau, nonlinear Schrödinger, non-Newtonian fluids and so on.

The Hilbert 19th Problem has been brilliantly solved by De Giorgi in [5] for $W_0^{1,2}$-weak solutions to *linear* differential operators over Lipschitz continuous domains when $m = 2$, $\varphi \in L^p$ with $p > n$ and $\psi \in L^q$ with $2q > n$, and this provided the initial breakthrough in the modern theory of quasilinear equations in more than two independent variables. The De Giorgi result was extended to *linear* equations in the *non-L^p* settings (i.e., when a sort of (1.2) holds) by Morrey in [18] and Lewy and Stampacchia in [16] to equations with measures at the right-hand side, assuming $\varphi \in L^{2,\lambda}$, $\psi \in L^{1,\mu}$ with $\lambda, \mu > n-2$. Moving to the quasilinear equation (1.1), we dispose of the seminal L^p-result of Serrin [25], which provides *interior* boundedness and Hölder continuity of the $W_0^{1,m}$-weak solutions to (1.1) in the *sub-controlled* case when the nonlinearities grow as $|u|^{m-1} + |Du|^{m-1}$, and the behaviour with respect to x of $\mathbf{a}(x,u,Du)$ and $b(x,u,Du)$ is controlled in terms of φ and ψ, respectively,

with

$$\tag{1.3} \begin{array}{lll} \varphi\in L^p(\Omega) & \text{with } p>\frac{m}{m-1}, & (m-1)p>n \\ \psi\in L^q(\Omega) & \text{with } q>\frac{mn}{mn+m-n}, & mq>n. \end{array}$$

Global boundedness of the $W_0^{1,m}$-weak solutions to (1.1) with general nonlinearities of *controlled* growths has been obtained by Ladyzhenskaya and Ural'tseva in [12] under the hypotheses (1.3) and for domains satisfying the *measure density (A)-condition.* Assuming *natural growths* of the data $\big($that is, $\mathbf{a}(x,u,Du)=\mathcal{O}(\varphi(x)+|Du|^{m-1})$ and $b(x,u,Du)=\mathcal{O}(\psi(x)+|Du|^m)\big)$ and (1.3), Ladyzhenskaya and Ural'tseva proved later in [13] Hölder continuity up to the boundary for the *bounded* weak solutions of (1.1), and Gariepy and Ziemer extended in [6] their result to domains with P*-thick complements.* It was Trudinger [26] the first to get *global* Hölder continuity of the *bounded* solutions in the *non-*L^p settings under the *natural structure hypotheses* of Ladyzhenskaya and Ural'tseva with $\varphi\in L^{n/(m-1),\varepsilon}$, $\psi\in L^{n/m,\varepsilon}$ for a small $\varepsilon>0$, while Lieberman derived in [17] a very general result on *interior* Hölder continuity when φ and ψ are suitable measures. We refer the author also to the works by Rakotoson [23], Rakotoson and Ziemer [24] and Zamboni [27] for various *interior* regularity results regarding the problem (1.1).

This paper is a natural continuation of [2] where boundedness has been proved for (1.1) with Morrey data in the case $m=2$ under the two-sided (A) condition on $\partial\Omega$. Here we derive *global boundedness* (Theorem 2.1) and *Hölder continuity up to the boundary* (Theorem 2.3) for *each* $W_0^{1,m}(\Omega)$-weak solution of the *coercive* Dirichlet problem (1.1) over domains with P*-thick* complements assuming *controlled growths* of the nonlinearities and Morrey data φ and ψ satisfying (1.2). Apart from the more general class of domains considered, we extend this way the classical L^p*-results* of Ladyzhenskaya and Ural'tseva [12, 13, 14] to the *non-*L^p*-settings* by weakening the hypotheses on φ and ψ to the scales of *Morrey type.* A comparison between (1.2) and (1.3) shows that the decrease of the degrees p and q of Lebesgue integrability of the data φ and ψ is at the expense of increase of the Morrey exponents λ and μ, and the range of these variations is always controlled by the relations $(m-1)p+\lambda>n$ and $mq+\mu>n$. Indeed, in the particular case $\lambda=\mu=0$ and domains with exterior cone property, our results reduce to these of Ladyzhenskaya and Uraltseva [12, 13, 14]. However, our Theorems 2.1 and 2.3 generalize substantially the results in [12, 13, 14] because even if $(m-1)p\leq n$ and $mq\leq n$, there exist functions $\varphi\in L^{p,\lambda}$ with $(m-1)p+\lambda>n$ and $\psi\in L^{q,\mu}$ with $mq+\mu>n$ for which (1.2) hold, but $\varphi\notin L^{p'}$ $\forall p'>n/(m-1)$ and $\psi\notin L^{q'}$ $\forall q'>n/m$ and therefore (1.3) fail. Moreover, as will be seen in Section 4 below, the *controlled* growths and the restrictions (1.2) on the Sobolev–Morrey exponents are *optimal* for the global boundedness and the subsequent Hölder continuity of the weak solutions to (1.1).

The paper is organized as follows. In Section 2 we start with introducing the concept of P-thickness and discuss its relations to the measure density property of $\partial\Omega$. We list in a detailed way the hypotheses imposed on the data of (1.1) and state the main results of the paper. Section 3 collects various auxiliary results which form the analytic heart of our approach. Of particular interest here is the Gehring–Giaquinta–Modica type Lemma 3.8 that asserts *better integrability* for the gradient of the weak solution over domains with P-thick complements, a particular case of which is due to Kilpeläinen and Koskela [11]. The proof of the *global boundedness* result (Theorem 2.1) is given in Section 4. Our technique relies on the

De Giorgi approach to the boundedness as adapted by Ladyzhenskaya and Uraltseva (cf. [14, Chapter IV]) to quasilinear equations. Namely, using the controlled growth assumptions, we get exact decay estimates for the total mass of the weak solution taken over its level sets. However, unlike the L^p-approach of Ladyzhenskaya and Uraltseva, the mass we have to do with is taken with respect to a positive Radon measure $\mathcal{M}$, which depends not only on the Lebesgue measure, but also on $\varphi^{\frac{m}{m-1}}$, ψ and a suitable power of the weak solution itself. Thanks to the hypotheses (1.2), the measure $\mathcal{M}$ allows to employ very precise inequalities of trace type due to D.R. Adams [1] and these lead to a bound of the $\mathcal{M}$-mass of u in terms of the m-energy of u. At this point we combine the controlled growth conditions with the better integrability of the gradient in order to estimate the m-energy of u in terms of small multiplier of the same quantity plus a suitable power of the level set $\mathcal{M}$-measure. The global boundedness of the weak solution then follows by a classical result known as *Hartman–Stampacchia maximum principle*. At the end of Section 4 we show sharpness of the controlled growths hypotheses as well as of (1.2) on the level of explicit examples built on quasilinear operators with m-Laplacean principal part. Section 5 is devoted to the proof of the *global Hölder continuity* as claimed in Theorem 2.3. Indeed, the boundedness of the weak solution is guaranteed by Theorem 2.1 and the fine results obtained by Lieberman in [17] apply to infer *interior Hölder continuity.* To extend it up to the boundary of Ω, we adopt to our situation the approach of Gariepy and Ziemer from [6] which relies on the Moser iteration technique in obtaining growth estimates for the gradient of the solution. The crucial step here is ensured by Lemma 5.1 which combines with the P-thickness condition in order to get estimate for the oscillation of u over small balls centered on $\partial\Omega$ in terms of a suitable positive power of the radius. Just for the sake of simplicity, we proved Theorem 2.3 under the controlled growths hypotheses. Following the same arguments, it is easy to see that the *global Hölder continuity* result still holds true for the *bounded* weak solutions of (1.1) if one assumes the *natural structure* conditions of Ladyzhenskaya and Ural'tseva instead of the controlled ones (cf. Theorem 5.2).

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2. Hypotheses and Main Results

Throughout the paper, we will use standard notations and will assume that the functions and sets considered are measurable.

We denote by $B_\rho(x)$ (or simply B_ρ if there is no ambiguity) the n-dimensional open ball with center $x \in \mathbb{R}^n$ and radius ρ. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ will be denoted by $|E|$ while, for any integrable function u defined on a set A, its integral average is given by

$$\overline{u}_A := \fint_A u(x)\,dx = \frac{1}{|A|}\int_A u(x)\,dx.$$

We will denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions over a bounded domain $\Omega \subset \mathbb{R}^n$ with compact support contained in that domain, and

$L^p(\Omega)$ stands for the standard Lebesgue space with a given $p \in [1,\infty]$. The Sobolev space $W_0^{1,p}(\Omega)$ is defined, as usual, by the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}$$

for $p \in [1,\infty)$.

Given $s \in [1,\infty)$ and $\theta \in [0,n]$, the Morrey space $L^{s,\theta}(\Omega)$ is the collection of all functions $u \in L^s(\Omega)$ such that

$$\|u\|_{L^{s,\theta}(\Omega)} := \sup_{x_0\in\Omega,\ \rho>0} \left(\rho^{-\theta} \int_{B_\rho(x_0)\cap\Omega} |u(x)|^s\, dx\right)^{1/s} < \infty.$$

The space $L^{s,\theta}(\Omega)$, equipped with the norm $\|\cdot\|_{L^{s,\theta}(\Omega)}$ is Banach space and the limit cases $\theta = 0$ and $\theta = n$ give rise, respectively, to $L^s(\Omega)$ and $L^\infty(\Omega)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $n \geq 2$. In order to set down the requirements on $\partial\Omega$, we need to recall the concept of *variational p-capacity* of a set for $1 < p < \infty$. Thus, given a compact set $C \subset \Omega$, its p-capacity is defined as

$$\mathrm{Cap}_p(C,\Omega) = \inf_g \int_\Omega |Dg|^p\, dx$$

where the infimum is taken over all functions $g \in C_0^\infty(\Omega)$ such that $g = 1$ in C. If $U \subset \Omega$ is an open set, then

$$\mathrm{Cap}_p(U,\Omega) = \sup_{C\subset U} \mathrm{Cap}_p(C,\Omega), \quad C \text{ is compact},$$

while, if $E \subset \Omega$ is an arbitrary set, then

$$\mathrm{Cap}_p(E,\Omega) = \inf_{E\subset U\subset\Omega} \mathrm{Cap}_p(U,\Omega), \quad U \text{ is open}.$$

In particular, if $E \subset E' \subset \Omega' \subset \Omega$ then

$$\mathrm{Cap}_p(E,\Omega) \leq \mathrm{Cap}_p(E',\Omega')$$

and, in case of two concentric balls B_R and B_r with $R > r$, the next formula

$$\mathrm{Cap}_p(\overline{B}_r, B_R) = Cr^{n-p}$$

is known for $p > 1$, where $C > 0$ depends on n, p and R/r (see [9, Chapter 2] for more details).

In the sequel we will suppose that the complement $\mathbb{R}^n \setminus \Omega$ of Ω satisfies the next *uniform P-thickness condition* for some $P \in (1,m)$: *there exist positive constants A_Ω and r_0 such that*

$$\mathrm{Cap}_P\big((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_r(x), B_{2r}(x)\big) \geq A_\Omega\ \mathrm{Cap}_P\big(\overline{B}_r(x), B_{2r}(x)\big) \tag{2.1}$$

for all $x \in \mathbb{R}^n \setminus \Omega$ and all $r \in (0, r_0)$.

Let us point out that replacing the capacity above with the Lebesgue measure, (2.1) reduces to the measure density condition (the *(A)-condition* of Ladyzhenskaya and Ural'tseva) which holds for instance when Ω supports the *uniform exterior cone property.* If a given set E satisfies the measure density condition then it is uniformly p-thick for each $p > 1$, whereas each nonempty set is uniformly p-thick if $p > n$. Further on, a uniformly q-thick set is also uniformly p-thick for all $p \geq q$ and, as proved in [15], the uniformly p-thick sets have a deep self-improving property to be uniformly q-thick for *some* $q < p$, depending on n, p and the constant of the p-thickness. In this sense, it is not restrictive to ask $P < m$ in (2.1) since even

if $\mathbb{R}^n \setminus \Omega$ were m-thick, the existence of a $P < m$ verifying (2.1) is ensured by [15]. Yet another example of uniformly p-thick sets for all $p > 1$ is given by those satisfying the uniform *corkscrew* condition: a set E is uniformly corkscrew if there exist constants $C > 0$ and $r_0 > 0$ such that for any $x \in E$ and any $r \in (0, r_0)$ there is a point $y \in B_r(x) \setminus E$ with the property that $B_{r/C}(y) \subset \mathbb{R}^n \setminus E$.

Turning back to the Dirichlet problem (1.1), the nonlinearities considered are given by the Carathéodory maps $\mathbf{a}\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and $b\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, where $\mathbf{a}(x, z, \xi) = \big(a^1(x, z, \xi), \cdots , a^n(x, z, \xi)\big)$. In other words, the functions $a^i(x, z, \xi)$ and $b(x, z, \xi)$ are measurable with respect to $x \in \Omega$ for all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and are continuous with respect to $z \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ for almost all (a.a.) $x \in \Omega$. Moreover, we suppose:

- *Controlled growth conditions:* There exist a constant $\Lambda > 0$ and non-negative functions $\varphi \in L^{p,\lambda}(\Omega)$ with $p > \frac{m}{m-1}$, $\lambda \in (0, n)$ and $(m - 1)p + \lambda > n$, and $\psi \in L^{q,\mu}(\Omega)$ with $q > \frac{mn}{mn+m-n}$, $\mu \in (0, n)$ and $mq + \mu > n$, such that

$$\begin{cases} |\mathbf{a}(x, z, \xi)| \leq \Lambda \left(\varphi(x) + |z|^{\frac{\ell(m-1)}{m}} + |\xi|^{m-1}\right), \\ |b(x, z, \xi)| \leq \Lambda \left(\psi(x) + |z|^{\ell-1} + |\xi|^{\frac{m(\ell-1)}{\ell}}\right) \end{cases} \tag{2.2}$$

for a.a. $x \in \Omega$ and all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$. Here, ℓ is the Sobolev conjugate of m and is given by

$$\ell = \begin{cases} \frac{nm}{n-m} & \text{if } m < n, \\ \text{any exponent } \ell > n & \text{if } m = n. \end{cases} \tag{2.3}$$

- *Coercivity condition:* There exists a constant $\gamma > 0$ such that

$$\mathbf{a}(x, z, \xi) \cdot \xi \geq \gamma|\xi|^m - \Lambda|z|^\ell - \Lambda\varphi(x)^{\frac{m}{m-1}} \tag{2.4}$$

for a.a. $x \in \Omega$ and all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Recall that a function $u \in W_0^{1,m}(\Omega)$ is called *weak solution* to the Dirichlet problem (1.1) if

$$\int_\Omega \mathbf{a}(x, u(x), Du(x)) \cdot Dv(x)\, dx + \int_\Omega b(x, u(x), Du(x))v(x)\, dx = 0 \tag{2.5}$$

for each test function $v \in W_0^{1,m}(\Omega)$. It is worth noting that the convergence of the integrals involved in (2.5) for all admissible u and v is ensured by (2.2) under the *sole* assumptions $p \geq \frac{m}{m-1}$ and $q \geq \frac{mn}{mn+m-n}$ when $m < n$, $q > 1$ if $m = n$.

Throughout the paper the omnibus phrase "*known quantities*" means that a given constant depends on the data in hypotheses (2.1)–(2.4), which include n, m, ℓ, γ, Λ, p, λ, q, μ, $\|\varphi\|_{L^{p,\lambda}(\Omega)}$, $\|\psi\|_{L^{q,\mu}(\Omega)}$, P, $\operatorname{diam}\Omega$, A_Ω and r_0. We will denote by C a generic constant, depending on known quantities, which may vary within the same formula.

Our first result claims *global essential boundedness* of the weak solutions to the problem (1.1).

Theorem 2.1. *Let* Ω *satisfy* (2.1) *and assume* (2.2) *and* (2.4). *Then each* $W_0^{1,m}(\Omega)$-*weak solution to the problem* (1.1) *is globally essentially bounded. That is, there exists a constant M, depending on known quantities, on* $\|Du\|_{L^m(\Omega)}$ *and on the uniform integrability of* $|Du|^m$, *such that*

$$\|u\|_{L^\infty(\Omega)} \leq M. \tag{2.6}$$

An immediate consequence of Theorem 2.1 and the local properties of solutions to quasilinear elliptic equations (cf. [17, 27]) is the *interior Hölder continuity* of the weak solutions.

Corollary 2.2. *Under the hypotheses of Theorem 2.1, each weak solution to* (1.1) *is* locally *Hölder continuous in* Ω. *That is,*

$$\sup_{x,y\in\Omega',\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq H \quad \forall \Omega' \Subset \Omega$$

with an exponent $\alpha \in (0,1)$ *and a constant* $H > 0$ *depending on the same quantities as* M *in* (2.6) *and on* $\operatorname{dist}(\Omega', \partial\Omega)$ *in addition.*

What really turns out is that assumptions (2.1), (2.2) and (2.4) are also sufficient to ensure Hölder continuity of the weak solutions *up to the boundary*, and this is the essence of our second main result.

Theorem 2.3. *Assume* (2.1)*,* (2.2) *and* (2.4)*. Then each weak solution of the Dirichlet problem* (1.1) *is* globally *Hölder continuous in* Ω. *Precisely,*

$$\sup_{x,y\in\overline{\Omega},\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq H,$$

where the exponent $\alpha \in (0,1)$ *and the Hölder constant* $H > 0$ *depend on the same quantities as* M *in* (2.6)*.*

3. Auxiliary Results

For the sake of completeness, we collect here some auxiliary results to be used in proving Theorems 2.1 and 2.3.

3.1. Basic tools.

Proposition 3.1. (Embeddings between Morrey spaces, see [22]) *For arbitrary* $s', s'' \in [1,\infty)$ *and* $\theta', \theta'' \in [0,n)$*, one has*

$$L^{s',\theta'}(\Omega) \subseteq L^{s'',\theta''}(\Omega)$$

if and only if

$$s' \geq s'' \geq 1 \quad \text{and} \quad \frac{s'}{n-\theta'} \geq \frac{s''}{n-\theta''}.$$

Proposition 3.2. (Hartman–Stampacchia maximum principle, see [10], [14]*Chapter II, Lemma 5.1) *Let* $\tau\colon \mathbb{R} \to [0,\infty)$ *be a non-increasing function and suppose there exist constants* $C > 0$, $k_0 \geq 0$, $\delta > 0$ *and* $\alpha \in [0, 1+\delta]$ *such that*

$$\int_k^\infty \tau(t)\, dt \leq Ck^\alpha \big(\tau(k)\big)^{1+\delta} \quad \forall k \geq k_0.$$

Then τ *supports the* finite time extinction property*, that is, there is a number* $k_{\max}$*, depending on* C*,* k_0*,* δ*,* α *and* $\int_{k_0}^\infty \tau(t)\, dt$*, such that*

$$\tau(k) = 0 \quad \forall k \geq k_{\max}.$$

Proposition 3.3. (Adams trace inequality, see [1]) *Let* $\mathcal{M}$ *be a positive Radon measure supported in* Ω *and such that* $\mathcal{M}(B_\rho(x)) \leq K\rho^{\alpha_0}$ *for each* $x \in \mathbb{R}^n$ *and each* $\rho > 0$*, where* K *is an absolute constant and*

$$\alpha_0 = \frac{s}{r}(n-r), \quad 1 < r < s < \infty, \quad r < n.$$

Then

$$\left(\int_\Omega |v(x)|^s \, d\mathcal{M}\right)^{1/s} \leq C(n,s,r)K^{1/s}\left(\int_\Omega |Dv(x)|^r \, dx\right)^{1/r} \qquad \forall v \in W_0^{1,r}(\Omega).$$

In particular, if $d\mathcal{M} = c(x)\,dx$ *with* $c \in L^{1,n-r+\varepsilon_0}(\Omega)$ *and* $\varepsilon_0 > 0$, *then*

$$\left(\int_\Omega |v(x)|^s c(x)\, dx\right)^{1/s} \leq C\left(n,s,r,\|c\|_{L^{1,n-r+\varepsilon_0}(\Omega)}\right)\left(\int_\Omega |Dv(x)|^r \, dx\right)^{1/r}$$

for all $v \in W_0^{1,r}(\Omega)$, *where* $n-r+\varepsilon_0 = \frac{s}{r}(n-r)$, $1<r<s<\infty$, $r<n$.

Proposition 3.4. (Gehring–Giaquinta–Modica lemma, see [7, Proposition 1.1, Chapter V]) *Let* B *be a fixed ball and* $G \in L^s(B)$, $F \in L^{s_0}(B)$ *be nonnegative functions with* $s_0 > s > 1$. *Suppose*

$$⨍_{B_\rho} G^s(x)\,dx \leq c\left(⨍_{B_{2\rho}} G(x)\,dx\right)^s + ⨍_{B_{2\rho}} F^s(x)\,dx + \theta ⨍_{B_{2\rho}} G^s(x)\,dx$$

for each ball B_ρ *of radius* $\rho \in (0,\rho_0)$ *such that* $B_{2\rho} \subset B$, *where* $0 \leq \theta < 1$.

Then there exist constants C *and* $m_0 \in (s, s_0]$, *depending on* n, c, s, s_0 *and* θ, *such that*

$$\left(⨍_{B_\rho} G^{m_0}(x)\,dx\right)^{1/m_0} \leq C\left(\left(⨍_{B_{2\rho}} G^s(x)\,dx\right)^{1/s} + \left(⨍_{B_{2\rho}} F^{s_0}(x)\,dx\right)^{1/s_0}\right).$$

Proposition 3.5. (John–Nirenberg lemma, see [26, Lemma 1.2], [8, Theorem 7.21]) *Let* B_0 *be a ball in* $\mathbb{R}^n$, $u \in W^{1,m}(B_0)$ *and suppose that, for any ball* $B \subset B_0$ *with the same center as* B_0 *there exists a constant* K *such that*

$$\|Du\|_{L^m(B)} \leq K|B|^{\frac{n-m}{mn}}.$$

Then there exists constants $\sigma_0 > 0$ *and* C *depending on* K, m, n *such that*

$$\int_{B_0} e^{\sigma_0 u}\,dx \int_{B_0} e^{-\sigma_0 u}\,dx \leq C|B_0|^2.$$

Proposition 3.6. (see [8, Lemma 8.23]) *Let* F *and* G *be nondecreasing functions in an interval* $(0,R]$. *Suppose that for all* $\rho \leq R$ *one has*

$$G(\rho/2) \leq c_0\big(G(\rho) + F(\rho)\big)$$

for some $0 < c_0 < 1$. *Then for any* $0<\tau<1$ *and* $\rho \leq R$ *we have*

$$G(\rho) \leq C\left(\left(\frac{\rho}{R}\right)^\alpha G(R) + F(\rho^\tau R^{1-\tau})\right)$$

where $C = C(c_0)$ *and* $\alpha = \alpha(c_0,\tau)$ *are positive constants.*

3.2. Boundary Sobolev inequality. The next result is a boundary variant of the Sobolev inequality which holds under the P-thickness condition.

Lemma 3.7. (Boundary Sobolev inequality) *Let* Ω *be a bounded domain with uniformly* P*-thick complement* $\mathbb{R}^n \setminus \Omega$ *and consider a function* $u \in W_0^{1,m}(\Omega)$ *which is extended as zero outside* Ω.

Let B_ρ *be a ball of radius* $\rho \in (0, r_0/(1-\theta))$, *centered at a point of* Ω *and suppose* $B_{\theta\rho} \setminus \Omega \neq \emptyset$ *for some* $0<\theta<1$.

Then for any $s \in [P, m]$ there is a constant $C = C(n, \theta, s, P, A_\Omega)$ such that

$$\left(\fint_{B_\rho} |u(x)|^{\tilde{s}}\, dx \right)^{1/\tilde{s}} \leq C\rho \left(\fint_{B_\rho} |Du(x)|^{s}\, dx \right)^{1/s} \tag{3.1}$$

for each $\tilde{s} \in [s, s^*]$, where s^* is the Sobolev conjugate of s ($s^* = ns/(n-s)$ if $s < n$ and s^* is any exponent greater than n otherwise).

Proof. Without loss of generality we may suppose that u is an s-quasicontinuous function in $W^{1,s}(B_\rho)$. Since $B_{\theta\rho} \setminus \Omega \neq \emptyset$, we can take a ball $B_{(1-\theta)\rho}(x_0)$ of radius $(1-\theta)\rho$, centered at $x_0 \in \partial\Omega$ and such that $B_{(1-\theta)\rho}(x_0) \subset B_\rho$. Setting $\mathcal{N}(u) = \{x \in B_\rho \colon u(x) = 0\}$ and applying the Hölder inequality and [11, Lemma 3.1], we get

$$\begin{aligned} \left(\fint_{B_\rho} |u(x)|^{\tilde{s}}\, dx \right)^{1/\tilde{s}} &\leq \left(\fint_{B_\rho} |u(x)|^{s^*}\, dx \right)^{1/s^*} \\ &\leq C \left(\frac{1}{\mathrm{Cap}_s\big(\mathcal{N}(u), B_{2\rho}\big)} \int_{B_\rho} |Du(x)|^{s}\, dx \right)^{1/s} \end{aligned} \tag{3.2}$$

whenever $s < n$. Indeed, (3.2) holds also for any $s^* > n$ when $s = n$. In fact,

$$\mathrm{Cap}_{s'}\big(\mathcal{N}(u), B_{2\rho}\big) \leq C\rho^{n-s'} \mathrm{Cap}_n\big(\mathcal{N}(u), B_{2\rho}\big)^{s'/n}$$

for abtrary $s' < n$, whence we have

$$\left(\frac{1}{\mathrm{Cap}_{s'}\big(\mathcal{N}(u), B_{2\rho}\big)} \int_{B_\rho} |Du|^{s'}\, dx \right)^{1/s'} \leq \left(\frac{1}{\mathrm{Cap}_{n}\big(\mathcal{N}(u), B_{2\rho}\big)} \int_{B_\rho} |Du|^{n}\, dx \right)^{1/n}.$$

Taking $s' = \frac{ns^*}{n+s^*} < n$ in (3.2) and using the above inequality, we get (3.2) for $s = n$ and for arbitrary $s^* > n$.

Since $u = 0$ in $\mathbb{R}^n \setminus \Omega$ except of a set of s-capacity zero and $B_{(1-\theta)\rho}(x_0) \subset B_\rho$, we have

$$\mathrm{Cap}_s\big(\mathcal{N}(u), B_{2\rho}\big) \geq \mathrm{Cap}_s\big(B_\rho \setminus \Omega, B_{2\rho}\big) \geq \mathrm{Cap}_s\big(B_{(1-\theta)\rho}(x_0) \setminus \Omega, B_{2\rho}\big)$$

by the properties of capacity, whereas

$$\mathrm{Cap}_s\big(B_{(1-\theta)\rho}(x_0) \setminus \Omega, B_{2\rho}\big) \geq C(n,\theta,s)\mathrm{Cap}_s\big(B_{(1-\theta)\rho}(x_0) \setminus \Omega, B_{2(1-\theta)\rho}(x_0)\big). \tag{3.3}$$

In fact, to see (3.3) we take functions $v \in C_0^\infty(B_{2\rho})$, $0 \leq v \leq 1$, $v = 1$ on $B_{(1-\theta)\rho}(x_0) \setminus \Omega$ and $\eta \in C_0^\infty(B_{2(1-\theta)\rho}(x_0))$, $|D\eta| \leq \frac{c}{(1-\theta)\rho}$, $0 \leq \eta \leq 1$, $\eta = 1$ on $B_{(1-\theta)\rho}(x_0)$. Then $v\eta \in C_0^\infty(B_{2(1-\theta)\rho}(x_0))$, $0 \leq v\eta \leq 1$, $v\eta = 1$ on $B_{(1-\theta)\rho}(x_0) \setminus \Omega$

and therefore, if $s<n$, we have

$$
\begin{aligned}
(3.4)\qquad & \operatorname{Cap}_s\big(B_{(1-\theta)\rho}(x_0)\setminus\Omega, B_{2(1-\theta)\rho}(x_0)\big)\\
&\le \int_{B_{2(1-\theta)\rho}(x_0)} |D(v\eta)|^s\,dx\\
&\le C\left(\int_{B_{2(1-\theta)\rho}(x_0)} |Dv|^s\,dx + \left(\frac{1}{(1-\theta)\rho)}\right)^s \int_{B_{2(1-\theta)\rho}(x_0)} |v|^s\,dx\right)\\
&\le C\left(\int_{B_{2\rho}} |Dv|^s\,dx + \left(\int_{B_{2\rho}} |v|^{s^*}dx\right)^{s/s^*}\right)\\
&\le C\int_{B_{2\rho}} |Dv|^s\,dx.
\end{aligned}
$$

The same bound holds true also if $s=n$ with a constant C, depending on θ in addition. Actually, making use of the Hölder and Sobolev ([9, 15.30]) inequalities, for arbitrary $t>1$ we get

$$
\begin{aligned}
\left(\frac{1}{(1-\theta)\rho)}\right)^n \int_{B_{2(1-\theta)\rho}(x_0)} |v|^n\,dx &\le \left(\frac{1}{(1-\theta)\rho)}\right)^n \left(\int_{B_{2\rho}} |v|^{nt}\,dx\right)^{1/t} |B_{2\rho}|^{1-1/t}\\
&\le \frac{1}{(1-\theta)^n}\rho^{-\frac{n}{t}} \left(\int_{B_{2\rho}} |v|^{nt}\,dx\right)^{1/t} = \frac{1}{(1-\theta)^n} \left(\fint_{B_{2\rho}} |v|^{nt}\,dx\right)^{1/t}\\
&\le \frac{C}{(1-\theta)^n}\rho^n \fint_{B_{2\rho}} |Dv|^n\,dx = \frac{C}{(1-\theta)^n}\int_{B_{2\rho}} |Dv|^n\,dx
\end{aligned}
$$

and thus (3.4) with $s=n$.

This way, (3.3) follows after taking the infimum in the right-hand side of (3.4) over all $v\in C_0^\infty(B_{2\rho})$ such that $v=1$ in $B_{(1-\theta)\rho}(x_0)\setminus\Omega$.

Further on, the uniform s-thickness condition (2.1) yields

$$
\begin{aligned}
&\operatorname{Cap}_s\big(B_{(1-\theta)\rho}(x_0)\setminus\Omega, B_{2(1-\theta)\rho}(x_0)\big)\\
&\qquad \ge C(n,s,P,A_\Omega)\operatorname{Cap}_s\big(B_{(1-\theta)\rho}(x_0), B_{2(1-\theta)\rho}(x_0)\big)\\
&\qquad = C(n,\theta,s,P,A_\Omega)\rho^{n-s}
\end{aligned}
$$

and therefore the desired estimate (3.1) follows from (3.3) and (3.2). □

3.3. Higher integrability of the gradient. The next result provides a crucial step to obtain global boundedness of the weak solutions to (1.1) although it is interesting by its own. Actually, it shows that the gradient of the weak solution to controlled growths and coercive problems (1.1) gains better integrability over domains with P-thick complements.

Lemma 3.8. *Assume* (2.1), (2.2) *and* (2.4), *and let* $u\in W_0^{1,m}(\Omega)$ *be a weak solution to the Dirichlet problem* (1.1).

Then there exist exponents $m_0>m$ *and* $\ell_0>\ell$ *such that* $u\in W^{1,m_0}(\Omega)\cap L^{\ell_0}(\Omega)$ *and*

$$\|Du\|_{L^{m_0}(\Omega)} + \|u\|_{L^{\ell_0}(\Omega)} \le C \tag{3.5}$$

with a constant C *depending on known quantities, on* $\|Du\|_{L^m(\Omega)}$ *and on the uniform integrability of* $|Du|^m$ *in* Ω.

Proof. Without loss of generality, we assume that the solution u and the data φ and ψ are extended as zero outside Ω. Let $x_0 \in \Omega$ be an arbitrary point and consider the concentric balls $B_\rho \subset B_{2\rho}$ centered at x_0 with $2\rho \in (0, r_0)$.

Case 1: $B_{3\rho/2} \setminus \Omega = \emptyset$. We have $B_{3\rho/2} \subset \Omega$ and let $\zeta \in C_0^\infty(B_{3\rho/2})$ be a cut-off function with the properties $0 \le \zeta \le 1$, $\zeta \equiv 1$ on B_ρ and $|D\zeta| \le c/\rho$. Employing (2.5) with a test function $v(x) = \left(u(x) - \overline{u}_{B_{3\rho/2}}\right)\zeta^2(x)$, we get

$$\begin{aligned}\int_\Omega a^i(x, u(x), Du(x))\left(\zeta^2(x)D_i u(x) + 2\left(u(x) - \overline{u}_{B_{3\rho/2}}\right)\zeta(x)D_i\zeta(x)\right)\,dx&\\ + \int_\Omega b(x, u(x), Du(x))\left(u(x) - \overline{u}_{B_{3\rho/2}}\right)\zeta^2(x)\,dx = 0.&\end{aligned}$$

Thus (2.2), (2.4) and the choice of ζ lead to

$$\begin{aligned}\gamma\int_{B_\rho}|Du(x)|^m\,dx \le{}& \Lambda\int_{B_{3\rho/2}}\varphi(x)^{\frac{m}{m-1}}\,dx + \Lambda\underbrace{\int_{B_{3\rho/2}}|u(x)|^\ell\,dx}_{I_1}\\ &+ 2\Lambda\underbrace{\int_{B_{3\rho/2}}\varphi(x)|u(x) - \overline{u}_{B_{3\rho/2}}||D\zeta(x)|\,dx}_{I_2}\\ &+ 2\Lambda\underbrace{\int_{B_{3\rho/2}}|u(x)|^{\frac{\ell(m-1)}{m}}|u(x) - \overline{u}_{B_{3\rho/2}}||D\zeta(x)|\,dx}_{I_3}\\ &+ 2\Lambda\underbrace{\int_{B_{3\rho/2}}|Du(x)|^{m-1}|u(x) - \overline{u}_{B_{3\rho/2}}||D\zeta(x)|\,dx}_{I_4}\\ &+ \Lambda\underbrace{\int_{B_{3\rho/2}}\psi(x)|u(x) - \overline{u}_{B_{3\rho/2}}|\,dx}_{I_5}\\ &+ \Lambda\underbrace{\int_{B_{3\rho/2}}|u(x)|^{\ell-1}|u(x) - \overline{u}_{B_{3\rho/2}}|\,dx}_{I_6}\\ &+ \Lambda\underbrace{\int_{B_{3\rho/2}}|Du(x)|^{\frac{m(\ell-1)}{\ell}}|u(x) - \overline{u}_{B_{3\rho/2}}|\,dx}_{I_7}.\end{aligned}\tag{3.6}$$

It follows from the triangle inequality that

$$I_1 = \int_{B_{3\rho/2}}|u(x)|^\ell\,dx \le C\int_{B_{3\rho/2}}|u(x) - \overline{u}_{B_{3\rho/2}}|^\ell\,dx + C\int_{B_{3\rho/2}}|\overline{u}_{B_{3\rho/2}}|^\ell\,dx.$$

In view of the Sobolev–Poincaré inequality, we get

$$\begin{aligned}\int_{B_{3\rho/2}} |u(x)-\overline{u}_{B_{3\rho/2}}|^\ell\, dx &\le C\rho^{\ell\left(n\left(\frac{1}{\ell}-\frac{1}{m}\right)+1\right)}\left(\int_{B_{3\rho/2}} |Du(x)|^m\, dx\right)^{\ell/m}\\ &= C\rho^{\ell\left(n\left(\frac{1}{\ell}-\frac{1}{m}\right)+1\right)}\left(\int_{B_{3\rho/2}} |Du(x)|^m\, dx\right)^{\ell/m-1}\int_{B_{3\rho/2}} |Du(x)|^m\, dx,\end{aligned}$$

while the Hölder inequality implies

$$\int_{B_{3\rho/2}} |\overline{u}_{B_{3\rho/2}}|^\ell\, dx = |B_{3\rho/2}|\left|\fint_{B_{3\rho/2}} u(x)\, dx\right|^\ell \le |B_{3\rho/2}|\left(\fint_{B_{3\rho/2}} |u(x)|^{\ell\frac{\widehat{m}}{m}}\, dx\right)^{m/\widehat{m}}$$

where $\widehat{m} := \max\left\{\frac{nm}{n+m}, 1\right\}$. Hence, the term I_1 is estimated as follows

$$\begin{aligned}I_1 \le{}& C\rho^{\ell\left(n\left(\frac{1}{\ell}-\frac{1}{m}\right)+1\right)}\left(\int_{B_{3\rho/2}} |Du(x)|^m\, dx\right)^{\ell/m-1}\int_{B_{3\rho/2}} |Du(x)|^m\, dx\\ &+ C|B_{3\rho/2}|\left(\fint_{B_{3\rho/2}} |u(x)|^{\ell\frac{\widehat{m}}{m}}\, dx\right)^{m/\widehat{m}}.\end{aligned}$$

Using the Young and the Sobolev–Poincaré inequalities, as well as $|D\zeta| \le c/\rho$, we get the bound

$$\begin{aligned}I_2 &= \int_{B_{3\rho/2}} \varphi(x)|u(x)-\overline{u}_{B_{3\rho/2}}||D\zeta(x)|\, dx\\ &\le C\int_{B_{3\rho/2}} |u(x)-\overline{u}_{B_{3\rho/2}}|^m|D\zeta(x)|^m\, dx + C\int_{B_{3\rho/2}} \varphi(x)^{\frac{m}{m-1}}\, dx\\ &\le \frac{C}{\rho^m}\int_{B_{3\rho/2}} |u(x)-\overline{u}_{B_{3\rho/2}}|^m\, dx + C\int_{B_{3\rho/2}} \varphi(x)^{\frac{m}{m-1}}\, dx\\ &\le C|B_{3\rho/2}|\left(\fint_{B_{3\rho/2}} |Du(x)|^{\widehat{m}}\, dx\right)^{m\widehat{m}} + C\int_{B_{3\rho/2}} \varphi(x)^{\frac{m}{m-1}}\, dx.\end{aligned}$$

In a similar manner one has

$$\begin{aligned}I_3 &= \int_{B_{3\rho/2}} |u(x)|^{\frac{\ell(m-1)}{m}}|u(x)-\overline{u}_{B_{3\rho/2}}||D\zeta(x)|\, dx\\ &\le C|B_{3\rho/2}|\left(\fint_{B_{3\rho/2}} |Du(x)|^{\widehat{m}}\, dx\right)^{m/\widehat{m}} + C\int_{B_{3\rho/2}} |u(x)|^\ell\, dx\\ &= C|B_{3\rho/2}|\left(\fint_{B_{3\rho/2}} |Du(x)|^{\widehat{m}}\, dx\right)^{m/\widehat{m}} + CI_1\end{aligned}$$

and

$$
\begin{aligned}
I_4 &= \int_{B_{3\rho/2}} |Du(x)|^{m-1}|u(x)-\overline{u}_{B_{3\rho/2}}||D\zeta(x)|\,dx \\
&\le C(\varepsilon)|B_{3\rho/2}|\left(\fint_{B_{3\rho/2}} |Du(x)|^{\widehat{m}}\,dx\right)^{m/\widehat{m}} + \varepsilon\int_{B_{3\rho/2}} |Du(x)|^m\,dx
\end{aligned}
$$

with arbitrary $\varepsilon > 0$.

To go further, we take $t = \ell = \frac{nm}{n-m}$ if $m < n$ and any $t > \max\{\frac{q}{q-1}, m\}$ otherwise, and apply successively the Hölder, Sobolev–Poincaré and Young inequalities. Thus, the following bound

$$
\begin{aligned}
I_5 &= \int_{B_{3\rho/2}} \psi(x)|u(x)-\overline{u}_{B_{3\rho/2}}|\,dx \\
&\le \left(\int_{B_{3\rho/2}} |u(x)-\overline{u}_{B_{3\rho/2}}|^t\,dx\right)^{1/t}\left(\int_{B_{3\rho/2}} \psi^{\frac{t}{t-1}}(x)\,dx\right)^{1-1/t} \\
&\le C\rho^{n\left(\frac{1}{t}-\frac{1}{m}\right)+1}\left(\int_{B_{3\rho/2}} |Du(x)|^m\,dx\right)^{1/m}\left(\int_{B_{3\rho/2}} \psi^{\frac{t}{t-1}}(x)\,dx\right)^{1-1/t} \\
&\le \varepsilon\int_{B_{3\rho/2}} |Du(x)|^m\,dx \\
&\quad + C(\varepsilon)\rho^{\frac{m}{m-1}\left(n\left(\frac{1}{t}-\frac{1}{m}\right)+1\right)}\left(\int_{B_{3\rho/2}} \psi^{\frac{t}{t-1}}(x)\,dx\right)^{(m/(m-1))(1-1/t)}
\end{aligned}
$$

holds true with an arbitrary $\varepsilon > 0$. In the same manner, I_6 and I_7 are estimated as well. Namely,

$$
\begin{aligned}
I_6 &= \int_{B_{3\rho/2}} |u(x)|^{\ell-1}|u(x)-\overline{u}_{B_{3\rho/2}}|\,dx \\
&\le C\int_{B_{3\rho/2}} |u(x)-\overline{u}_{B_{3\rho/2}}|^{\ell}\,dx + C\int_{B_{3\rho/2}} |u(x)|^{\ell}\,dx \\
&\le C\rho^{\ell\left(n\left(\frac{1}{\ell}-\frac{1}{m}\right)+1\right)}\left(\int_{B_{3\rho/2}} |Du(x)|^m\,dx\right)^{\ell/m-1}\int_{B_{3\rho/2}} |Du(x)|^m\,dx \\
&\quad + C\int_{B_{3\rho/2}} |u(x)|^{\ell}\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_7 &= \int_{B_{3\rho/2}} |Du(x)|^{\frac{m(\ell-1)}{\ell}}|u(x)-\overline{u}_{B_{3\rho/2}}|\,dx \\
&\le \left(\int_{B_{3\rho/2}} |u(x)-\overline{u}_{B_{3\rho/2}}|^{\ell}\,dx\right)^{1/\ell}\left(\int_{B_{3\rho/2}} |Du(x)|^m\,dx\right)^{1-1/\ell} \\
&\le C\rho^{n\left(\frac{1}{\ell}-\frac{1}{m}\right)+1}\left(\int_{B_{3\rho/2}} |Du(x)|^m\,dx\right)^{1/m-1/\ell}\int_{B_{3\rho/2}} |Du(x)|^m\,dx.
\end{aligned}
$$

At this point we employ the above bounds into (3.6), divide the both sides by ρ^n and then use $B_{3\rho/2} \subset B_{2\rho}$ in order to get

$$
\begin{aligned}
(3.7)\quad & \fint_{B_\rho} |Du(x)|^m + |u(x)|^\ell \, dx \\
& \leq C\left(\fint_{B_{2\rho}} \left(|Du(x)|^m + |u(x)|^\ell\right)^{\frac{\widehat{m}}{m}} \, dx\right)^{m/\widehat{m}} + C\fint_{B_{2\rho}} \varphi(x)^{\frac{m}{m-1}} \, dx \\
& \quad + C(\varepsilon)\rho^{\frac{m}{m-1}\left(n\left(\frac{1}{t}-\frac{1}{m}\right)+1\right)}\left(\int_{B_{2\rho}} \psi^{\frac{t}{t-1}}(x) \, dx\right)^{(m/(m-1))(1-1/t)-1} \\
& \qquad \times \fint_{B_{2\rho}} \psi^{\frac{t}{t-1}}(x) \, dx \\
& \quad + C\left(\varepsilon + \rho^{\ell\left(n\left(\frac{1}{\ell}-\frac{1}{m}\right)+1\right)}\left(\int_{B_{2\rho}} |Du(x)|^m \, dx\right)^{\ell/m-1}\right. \\
& \qquad \left. + \rho^{n\left(\frac{1}{\ell}-\frac{1}{m}\right)+1}\left(\int_{B_{2\rho}} |Du(x)|^m \, dx\right)^{1/m-1/\ell}\right)\fint_{B_{2\rho}} |Du(x)|^m \, dx.
\end{aligned}
$$

Case 2: $B_{3\rho/2} \setminus \Omega \neq \emptyset$. Take $v(x) = u(x)\zeta^2(x)$ as test function in (2.5) with $\zeta \in C_0^\infty(B_{2\rho})$, $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on B_ρ and $|D\zeta| \leq c/\rho$, and use (2.2), (2.4) and the properties of ζ to get

$$
\begin{aligned}
(3.8)\qquad \gamma\int_{B_\rho} |Du(x)|^m \, dx \leq{}& \Lambda\int_{B_{2\rho}} \varphi(x)^{\frac{m}{m-1}} \, dx + 2\Lambda\underbrace{\int_{B_{2\rho}} |u(x)|^\ell \, dx}_{J_1} \\
& + 2\Lambda\underbrace{\int_{B_{2\rho}} \varphi(x)|u(x)||D\zeta(x)| \, dx}_{J_2} \\
& + 2\Lambda\underbrace{\int_{B_{2\rho}} |u(x)|^{\frac{\ell(m-1)}{m}+1}|D\zeta(x)| \, dx}_{J_3} \\
& + 2\Lambda\underbrace{\int_{B_{2\rho}} |Du(x)|^{m-1}|u(x)||D\zeta(x)| \, dx}_{J_4} \\
& + \Lambda\underbrace{\int_{B_{2\rho}} \psi(x)|u(x)| \, dx}_{J_5} \\
& + \Lambda\underbrace{\int_{B_{2\rho}} |Du(x)|^{\frac{m(\ell-1)}{\ell}}|u(x)| \, dx}_{J_6}.
\end{aligned}
$$

We will estimate the terms on the right-hand side of (3.8) by means of the boundary Sobolev inequality (3.1). Precisely,

$$
\begin{aligned}
J_1 = \int_{B_{2\rho}} |u(x)|^\ell \, dx &\le C\rho^{\ell\left(n\left(\frac{1}{\ell}-\frac{1}{m}\right)+1\right)} \left(\int_{B_{2\rho}} |Du(x)|^m \, dx\right)^{\ell/m} \\
&= C\rho^{\ell\left(n\left(\frac{1}{\ell}-\frac{1}{m}\right)+1\right)} \left(\int_{B_{2\rho}} |Du(x)|^m \, dx\right)^{\ell/m-1} \int_{B_{2\rho}} |Du(x)|^m \, dx.
\end{aligned}
$$

Using the Young inequality, (3.1) and taking into account $|D\zeta| \le c/\rho$, we have

$$
\begin{aligned}
J_2 &= \int_{B_{2\rho}} \varphi(x)|u(x)||D\zeta(x)| \, dx \\
&\le C\int_{B_{2\rho}} |u(x)|^m |D\zeta(x)|^m \, dx + C\int_{B_{2\rho}} \varphi(x)^{\frac{m}{m-1}} \, dx \\
&\le \frac{C}{\rho^m}\int_{B_{2\rho}} |u(x)|^m \, dx + C\int_{B_{2\rho}} \varphi(x)^{\frac{m}{m-1}} \, dx \\
&\le C\rho^{n-n\frac{m}{\widehat{m}}} \left(\int_{B_{2\rho}} |Du(x)|^{\widehat{m}} \, dx\right)^{m/\widehat{m}} + C\int_{B_{2\rho}} \varphi(x)^{\frac{m}{m-1}} \, dx
\end{aligned}
$$

where $\widehat{m} := \max\left\{\frac{nm}{n+m}, P\right\}$. Similarly,

$$
\begin{aligned}
J_3 &= \int_{B_{2\rho}} |u(x)|^{\frac{\ell(m-1)}{m}+1} |D\zeta(x)| \, dx \le \frac{C}{\rho^m}\int_{B_{2\rho}} |u(x)|^m \, dx + C\int_{B_{2\rho}} |u(x)|^\ell \, dx \\
&\le C\rho^{n-n\frac{m}{\widehat{m}}} \left(\int_{B_{2\rho}} |Du(x)|^{\widehat{m}} \, dx\right)^{m/\widehat{m}} + C\int_{B_{2\rho}} |u(x)|^\ell \, dx \\
&= C\rho^{n-n\frac{m}{\widehat{m}}} \left(\int_{B_{2\rho}} |Du(x)|^{\widehat{m}} \, dx\right)^{m/\widehat{m}} + CJ_1
\end{aligned}
$$

and

$$
\begin{aligned}
J_4 &= \int_{B_{2\rho}} |Du(x)|^{m-1}|u(x)||D\zeta(x)| \, dx \\
&\le \varepsilon\int_{B_{2\rho}} |Du(x)|^m \, dx + \frac{C(\varepsilon)}{\rho^m}\int_{B_{2\rho}} |u(x)|^m \, dx \\
&\le \varepsilon\int_{B_{2\rho}} |Du(x)|^m \, dx + C(\varepsilon)\rho^{n-n\frac{m}{\widehat{m}}} \left(\int_{B_{2\rho}} |Du(x)|^{\widehat{m}} \, dx\right)^{m/\widehat{m}}
\end{aligned}
$$

with arbitrary $\varepsilon > 0$.

To estimate J_5, we take the exponent t as above, namely $t = \ell$ if $m < n$ and $t > \max\{\frac{q}{q-1}, m\}$ otherwise, and apply the Hölder inequality, (3.1) and the Young

inequality. Thus

$$
\begin{aligned}
J_5 = \int_{B_{2\rho}} \psi(x)|u(x)|\,dx &\le \left(\int_{B_{2\rho}} |u(x)|^t\,dx\right)^{1/t} \left(\int_{B_{2\rho}} \psi^{\frac{t}{t-1}}(x)\,dx\right)^{1-1/t} \\
&\le C\rho^{n\left(\frac{1}{t}-\frac{1}{m}\right)+1} \left(\int_{B_{2\rho}} |Du(x)|^m\,dx\right)^{1/m} \left(\int_{B_{2\rho}} \psi^{\frac{t}{t-1}}(x)\,dx\right)^{1-1/t} \\
&\le \varepsilon \int_{B_{2\rho}} |Du(x)|^m\,dx + C(\varepsilon)\rho^{\frac{m}{m-1}\left(n\left(\frac{1}{t}-\frac{1}{m}\right)+1\right)} \left(\int_{B_{2\rho}} \psi^{\frac{t}{t-1}}(x)\,dx\right)^{(m/(m-1)(1-1/t)}
\end{aligned}
$$

holds with arbitrary $\varepsilon > 0$, and the remaining term J_6 is estimated in the same manner

$$
\begin{aligned}
J_6 &= \int_{B_{2\rho}} |Du(x)|^{\frac{m(\ell-1)}{\ell}} |u(x)|\,dx \\
&\le \left(\int_{B_{2\rho}} |u(x)|^\ell\,dx\right)^{1/\ell} \left(\int_{B_{2\rho}} |Du(x)|^m\,dx\right)^{1-1/\ell} \\
&\le C\rho^{n\left(\frac{1}{\ell}-\frac{1}{m}\right)+1} \left(\int_{B_{2\rho}} |Du(x)|^m\,dx\right)^{1/m-1/\ell} \int_{B_{2\rho}} |Du(x)|^m\,dx.
\end{aligned}
$$

Using the bounds for $J_1 - J_6$ in (3.8) leads once again to (3.7) with the difference that $\widehat{m} := \max\left\{\frac{nm}{n+m}, P\right\}$ now. However, a careful analysis of the estimates for the terms $I_1 - I_4$ above shows that these remain valid also with $\widehat{m} := \max\left\{\frac{nm}{n+m}, P\right\}$ because of $P < m$ and the fact that the Sobolev conjugate of P is anyway *greater than* m if $P > \frac{nm}{n+m}$.

Therefore, (3.7) holds true with $\widehat{m} := \max\left\{\frac{nm}{n+m}, P\right\}$ in the both cases considered above.

Looking at (3.7) we recall that $n\left(\frac{1}{\ell} - \frac{1}{m}\right) + 1 \ge 0$ and $\frac{\ell}{m} > 1$. Thus, thanks also to the absolute continuity of the Lebesgue integral, we can choose ε and ρ_0 so small that if $\rho < \rho_0$ then the multiplier of $\fint_{B_{2\rho}} |Du(x)|^m\,dx$ at the right-hand side of (3.7) becomes less than $1/2$.

To apply Proposition 3.4, we consider the functions

$$
G(x) = \begin{cases} \left(|Du(x)|^m + |u(x)|^\ell\right)^{\frac{\widehat{m}}{m}} & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega \end{cases}
$$

and

$$
F(x) = \begin{cases} \left(C(\varepsilon)\rho_0^{\frac{m}{m-1}\left(n\left(\frac{1}{t}-\frac{1}{m}\right)+1\right)} \left(\int_\Omega \psi^{\frac{t}{t-1}}(x)\,dx\right)^{\frac{m}{m-1}\left(1-\frac{1}{t}\right)-1} \psi^{\frac{t}{t-1}}(x) \right. \\ \qquad \left. + C\varphi(x)^{\frac{m}{m-1}} \right)^{\frac{\widehat{m}}{m}} & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega, \end{cases}
$$

and set $s = \frac{m}{\widehat{m}}$, $s_0 = \kappa\frac{m}{\widehat{m}}$ where $\kappa \in \left(1, \min\left\{\frac{p(m-1)}{m}, \frac{q(t-1)}{t}\right\}\right)$. It is worth noting that the existence of such κ is ensured by our hypotheses

$$p > \frac{m}{m-1}, \quad q > \frac{nm}{nm+m-n}, \quad t > \max\left\{\frac{q}{q-1}, m\right\}.$$

Moreover, $n\left(\frac{1}{t} - \frac{1}{m}\right) + 1 \geq 0$ and $\frac{m}{m-1}\left(1 - \frac{1}{t}\right) - 1 > 0$.

With these settings, the inequality (3.7) rewrites into

$$\fint_{B_\rho} G^s(x)\, dx \leq C \left(\fint_{B_{2\rho}} G(x)\, dx\right)^s + \fint_{B_{2\rho}} F^s(x)\, dx + \frac{1}{2}\fint_{B_{2\rho}} G^s(x)\, dx$$

for each ball B_ρ with $\rho < \rho_0$ such that $B_{2\rho} \subset B$, where B is a large enough ball containing the bounded domain Ω.

At this point Proposition 3.4 applies to ensure existence of exponents $m_0 > m$ and $\ell_0 > \ell$, and a constant C such that

$$\|Du\|_{L^{m_0}(B_\rho)} + \|u\|_{L^{\ell_0}(B_\rho)} \leq C \quad \forall \rho < \rho_0.$$

The desired estimate (3.5), with a constant C depending on known quantities, on $\|Du\|_{L^m(\Omega)}$ and the uniform integrability of $|Du|^m$, then follows by simple covering argument. □

4. Global Essential Boundedness

4.1. Proof of Theorem 2.1. Let us start with the case $m < n$. Consider the measure

$$d\mathcal{M} := \left(\chi(x) + \varphi(x)^{\frac{m}{m-1}} + \psi(x) + |u(x)|^{\frac{m^2}{n-m}}\right)\, dx$$

where $\chi(x)$ is the characteristic function of the domain Ω, dx is the Lebesgue measure and φ and ψ are supposed to be extended as zero outside Ω.

Given a ball B_ρ of radius ρ, we employ the assumptions on φ and ψ to get

$$\int_{B_\rho} \varphi(x)^{\frac{m}{m-1}}\, dx \leq \|\varphi\|^{\frac{m}{m-1}}_{L^{p,\lambda}(\Omega)} \rho^{n-\frac{m(n-\lambda)}{p(m-1)}} = \|\varphi\|^{\frac{m}{m-1}}_{L^{p,\lambda}(\Omega)} \rho^{n-m+\left(m-\frac{m(n-\lambda)}{p(m-1)}\right)}$$

$$\int_{B_\rho} \psi(x)\, dx \leq \|\psi\|_{L^{q,\mu}(\Omega)} \rho^{n-\frac{n-\mu}{q}} = \|\psi\|_{L^{q,\mu}(\Omega)} \rho^{n-m+\left(m-\frac{n-\mu}{q}\right)}$$

with $m - \frac{m(n-\lambda)}{p(m-1)} > 0$ and $m - \frac{n-\mu}{q} > 0$ as consequence of the hypotheses $(m-1)p + \lambda > n$ and $mq + \mu > n$.

Further on, $u \in L^{\ell_0}(\Omega)$ by (3.5) and therefore the Hölder inequality gives

$$\int_{B_\rho} |u(x)|^{\frac{m^2}{n-m}}\, dx \leq \|u\|^{\frac{m^2}{n-m}}_{L^{\ell_0}(\Omega)} \rho^{n-m+\frac{m\ell_0(n-m)-nm^2}{\ell_0(n-m)}} \tag{4.1}$$

with $\frac{m\ell_0(n-m)-nm^2}{\ell_0(n-m)} > 0$ because of $\ell_0 > \ell = \frac{nm}{n-m}$.

This way, setting

$$\varepsilon_0 := \min\left\{m - \frac{m(n-\lambda)}{p(m-1)}, m - \frac{n-\mu}{q}, \frac{m\ell_0(n-m)-nm^2}{\ell_0(n-m)}\right\} > 0$$

we get

$$\mathcal{M}(B_\rho) \leq K\rho^{n-m+\varepsilon_0}$$

with a constant K depending on known quantities.

For an arbitrary $k \geq 1$, we consider now the function

$$v(x) := \max\{u(x) - k, 0\}$$

and its upper zero-level set

$$\Omega_k := \big\{x \in \Omega\colon\ u(x) > k\big\}.$$

It is immediate that $v \equiv 0$ on $\Omega \setminus \Omega_k$ and $v \in W_0^{1,m}(\Omega)$.

The Hölder inequality gives

$$\int_\Omega v(x)\, d\mathcal{M} = \int_{\Omega_k} v(x)\, d\mathcal{M} \leq \left(\int_{\Omega_k} d\mathcal{M}\right)^{1-1/s} \left(\int_{\Omega_k} |v(x)|^s\, d\mathcal{M}\right)^{1/s},$$

whence, applying the Adams trace inequality (Proposition 3.3) with

$$\alpha_0 = n - m + \varepsilon_0, \quad s = \frac{m(n-m+\varepsilon_0)}{n-m}, \quad r = m,$$

we get

$$\int_\Omega v(x)\, d\mathcal{M} \leq (\mathcal{M}(\Omega_k))^{1-\frac{n-m}{m(n-m+\varepsilon_0)}} \left(\int_{\Omega_k} |Dv(x)|^m\, dx\right)^{1/m}. \tag{4.2}$$

To estimate the $L^m(\Omega_k)$-norm of the gradient Du above, we will apply (2.2) and (2.4). For, the Young inequality implies

$$|\xi|^{\frac{nm-n+m}{n}}|z| \leq \varepsilon|\xi|^m + C(\varepsilon)|z|^{\frac{nm}{n-m}}$$

so that the controlled growth assumptions (2.2) yield

$$b(x,z,\xi)z \leq |z||b(x,z,\xi)| \leq \Lambda\Big(\varepsilon|\xi|^m + C(\varepsilon)|z|^{\frac{nm}{n-m}} + |z|\psi(x)\Big)$$

for a.a. $x \in \Omega$, for all $(x,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and with arbitrary $\varepsilon > 0$ to be chosen later. In particular, keeping in mind

$$0 < \frac{u(x)-k}{u(x)} < 1 \quad \text{a.e. } \Omega_k,$$

we have

$$\begin{aligned} |b(x,u(x),Du(x))v(x)| &= |b(x,u(x),Du(x))u(x)|\frac{u(x)-k}{u(x)} \\ &\leq \Lambda\Big(\varepsilon|Du(x)|^m + C(\varepsilon)|u(x)|^{\frac{nm}{n-m}} + |u(x)|\psi(x)\Big) \end{aligned} \tag{4.3}$$

for a.a. $x \in \Omega_k$.

At this point, we employ $v \in W_0^{1,m}(\Omega)$ as test function in (2.5) and use $v \equiv 0$ on $\Omega \setminus \Omega_k$, $|Dv| = |Du|$ a.e. Ω_k and (2.4), in order to conclude that

$$\begin{aligned} \int_{\Omega_k} |Dv(x)|^m\, dx \leq C\Bigg(& \underbrace{\int_{\Omega_k} \varphi(x)^{\frac{m}{m-1}}\, dx}_{I_1} + \underbrace{\int_{\Omega_k} |u(x)|\psi(x)\, dx}_{I_2} \\ & + \underbrace{\int_{\Omega_k} |u(x)|^{\frac{nm}{n-m}}\, dx}_{I_3}\Bigg) \end{aligned} \tag{4.4}$$

after choosing appropriately ε in (4.3).

It is immediate that

$$I_1 = \int_{\Omega_k} \varphi(x)^{\frac{m}{m-1}}\, dx \le \mathcal{M}(\Omega_k). \tag{4.5}$$

Further on, we have

$$\begin{aligned} I_2 &= \int_{\Omega_k} |u(x)|\psi(x)\, dx = \int_{\Omega_k} |u(x)-k+k|\psi(x)\, dx \\ &\le \int_{\Omega_k} v(x)\psi(x)\, dx + k\int_{\Omega_k} \psi(x)\, dx \\ &\le \int_{\Omega_k} v(x)\psi(x)\, dx + k\mathcal{M}(\Omega_k). \end{aligned} \tag{4.6}$$

To estimate the first term on the right-hand side above, define the measure $d\overline{\mathcal{M}} := \psi(x)\, dx$. We have $\overline{\mathcal{M}}(B_\rho) \le C(n)\rho^{n-\frac{n-\mu}{q}}$ for each ball B_ρ because of $\psi \in L^{q,\mu}(\Omega)$ and therefore Proposition 3.3 can be applied with $s = \frac{m\left(n-\frac{n-\mu}{q}\right)}{n-m}$ and $r = m$. Namely,

$$\begin{aligned} \int_{\Omega_k} v(x)\psi(x)\, dx = \int_{\Omega_k} v(x)\, d\overline{\mathcal{M}} &\le \left(\int_{\Omega_k} d\overline{\mathcal{M}}\right)^{1-1/s} \left(\int_{\Omega_k} |v(x)|^s\, d\overline{\mathcal{M}}\right)^{1/s} \\ &\le \left(\overline{\mathcal{M}}(\Omega_k)\right)^{1-1/s} \left(\int_{\Omega_k} |Dv(x)|^m\, dx\right)^{1/m}. \end{aligned}$$

We use the Young inequality to estimate the last term above by

$$\varepsilon \int_{\Omega_k} |Dv(x)|^m\, dx + C(\varepsilon)\left(\overline{\mathcal{M}}(\Omega_k)\right)^{\frac{m}{m-1}\frac{s-1}{s}}$$

with arbitrary $\varepsilon > 0$. Moreover,

$$\left(\overline{\mathcal{M}}(\Omega_k)\right)^{\frac{m}{m-1}\frac{s-1}{s}} \le \left(\mathcal{M}(\Omega_k)\right)^{\frac{m}{m-1}\frac{s-1}{s}} \le \mathcal{M}(\Omega_k)\left(\mathcal{M}(\Omega)\right)^{\frac{m}{m-1}\frac{s-1}{s}-1}$$

and

$$\begin{aligned} \mathcal{M}(\Omega) = &\int_\Omega dx + \int_\Omega \varphi(x)^{\frac{m}{m-1}}\, dx + \int_\Omega \psi(x)\, dx + \int_\Omega |u(x)|^{\frac{m^2}{n-m}}\, dx \\ &\le |\Omega| + C(n,m,p,\lambda,q,\mu,\operatorname{diam}\Omega)\left(\|\varphi\|^{\frac{m}{m-1}}_{L^{p,\lambda}(\Omega)} + \|\psi\|_{L^{q,\mu}(\Omega)} + \|u\|^{\frac{m^2}{n-m}}_{L^{\ell_0}(\Omega)}\right). \end{aligned}$$

Thus, remembering (3.5), $\mathcal{M}(\Omega)$ is bounded in terms of known quantities and $\|Du\|_{L^m(\Omega)}$, whence

$$I_2 \le \varepsilon \int_{\Omega_k} |Dv(x)|^m\, dx + C(\varepsilon)k\mathcal{M}(\Omega_k) \tag{4.7}$$

with arbitrary $\varepsilon > 0$.

In the same manner we estimate also the last term I_3 of (4.4). Precisely,

$$\begin{aligned} I_3 &= \int_{\Omega_k} |u(x)|^{\frac{nm}{n-m}}\,dx = \int_{\Omega_k} |u(x)|^m |u(x)|^{\frac{m^2}{n-m}}\,dx \\ &= \int_{\Omega_k} |u(x)-k+k|^m |u(x)|^{\frac{m^2}{n-m}}\,dx \\ &\leq 2^{m-1}\left(\int_{\Omega_k} v^m(x)|u(x)|^{\frac{m^2}{n-m}}\,dx + k^m \int_{\Omega_k} |u(x)|^{\frac{m^2}{n-m}}\,dx\right) \\ &\leq 2^{m-1}\int_{\Omega_k} v^m(x)|u(x)|^{\frac{m^2}{n-m}}\,dx + 2^{m-1}k^m\mathcal{M}(\Omega_k). \end{aligned}$$

We will estimate the first term above with the aid of the Adams trace inequality. For this goal, note that (4.1) implies $|u|^{\frac{m^2}{n-m}} \in L^{1,\theta}(\Omega)$ with

$$\theta = n - m + \frac{m\ell_0(n-m) - nm^2}{\ell_0(n-m)} > n-m.$$

Therefore, there exists an $r' < m$, close enough to m, and such that

$$n - m < \frac{m}{r'}(n-r') < \theta.$$

We have then

$$n - r' + \frac{(n-r')(m-r')}{r'} < \theta$$

and Proposition 3.1 yields $|u|^{\frac{m^2}{n-m}} \in L^{1,n-r'+\frac{(n-r')(m-r')}{r'}}(\Omega)$. This way, Proposition 3.3 and the Hölder inequality give

$$\begin{aligned} \int_{\Omega_k} v^m(x)|u(x)|^{\frac{m^2}{n-m}}\,dx &\leq C\left(\int_{\Omega_k} |Dv(x)|^{r'}\,dx\right)^{m/r'} \\ &\leq C|\Omega_k|^{\frac{m}{r'}-1}\left(\int_{\Omega_k} |Dv(x)|^m\,dx\right) \end{aligned}$$

with C depending also on $\left\| |u|^{\frac{m^2}{n-m}} \right\|_{L^{1,\theta}(\Omega)}$ which is bounded in terms of $\|u\|_{L^{\ell_0}(\Omega)}$ (cf. (4.1) and (3.5)). Therefore,

$$I_3 \leq C\left(|\Omega_k|^{\frac{m}{r'}-1}\int_{\Omega_k} |Dv(x)|^m\,dx + k^m\mathcal{M}(\Omega_k)\right) \tag{4.8}$$

and putting (4.5), (4.7) and (4.8) together, (4.4) takes on the form

$$\int_{\Omega_k} |Dv(x)|^m\,dx \leq C\left(|\Omega_k|^{\frac{m}{r'}-1}\int_{\Omega_k} |Dv(x)|^m\,dx + k^m\mathcal{M}(\Omega_k)\right) \tag{4.9}$$

after choosing $\varepsilon > 0$ small enough and remembering $k \geq 1$.

We have further

$$k^{\frac{nm}{n-m}}|\Omega_k| \leq \int_{\Omega_k} |u(x)|^{\frac{nm}{n-m}}\,dx \leq \int_{\Omega} |u(x)|^{\frac{nm}{n-m}}\,dx \leq C\|Du\|^{\frac{nm}{n-m}}_{L^m(\Omega)}$$

and this means that if $k \geq k_0$ for large enough k_0, depending on known quantities and on $\|Du\|_{L^m(\Omega)}$, then the multiplier factor $C|\Omega_k|^{\frac{m}{r'}-1}$ on the right-hand side of (4.9) can be made less than 1/2. This way

$$\int_{\Omega_k} |Dv(x)|^m\,dx \leq Ck^m\mathcal{M}(\Omega_k) \qquad \forall\ k \geq k_0 \tag{4.10}$$

and then (4.2) becomes

$$\int_{\Omega_k} v(x)\, d\mathcal{M} \le Ck\big(\mathcal{M}(\Omega_k)\big)^{1+\frac{\varepsilon_0}{m(n-m+\varepsilon_0)}} \qquad \forall\, k \ge k_0 \tag{4.11}$$

with $\varepsilon_0 > 0$.

Employing the Cavalieri principle, we have

$$\int_{\Omega_k} v(x)\, d\mathcal{M} = \int_{\Omega_k} (u(x)-k)\, d\mathcal{M} = \int_k^\infty \mathcal{M}(\Omega_t)\, dt$$

and the setting $\tau(t) := \mathcal{M}(\Omega_t)$ rewrites (4.11) into

$$\int_k^\infty \tau(t)\, dt \le Ck(\tau(k))^{1+\delta} \qquad \forall k \ge k_0,\ \delta = \frac{\varepsilon_0}{m(n-m+\varepsilon_0)} > 0.$$

It remains to apply the Hartman–Stampacchia maximum principle (Proposition 3.2) to conclude

$$u(x) \le k_{\max} \qquad \text{a.e. } \Omega$$

where $k_{\max}$ depends on known quantities and on $\|Du\|_{L^m(\Omega)}$ in addition.

Repeating the above procedure with $-u(x)$ instead of $u(x)$, we get a bound from below for $u(x)$ which gives the desired estimate (2.6) when $m < n$.

The claim of Theorem 2.1 in the limit case $m = n$ can be easily obtained by adapting the above procedure to the new situation. Precisely, the controlled growth condition (2.2) for the term $b(x,z,\xi)$ and the coercivity condition (2.4) have now the form

$$|b(x,z,\xi)| \le \Lambda\left(\psi(x) + |z|^{\ell-1} + |\xi|^{\frac{n(\ell-1)}{\ell}}\right), \tag{4.12}$$

$$\mathbf{a}(x,z,\xi)\cdot\xi \ge \gamma|\xi|^n - \Lambda|z|^\ell - \Lambda\varphi(x)^{\frac{n}{n-1}}, \tag{4.13}$$

respectively, where $\ell > n$ is an *arbitrary* exponent (cf. (2.3)), $\varphi \in L^{p,\lambda}(\Omega)$ with $p > \frac{n}{n-1}$, $\lambda \in (0,n)$ and $(n-1)p+\lambda > n$, $\psi \in L^{q,\mu}(\Omega)$ with $q > 1$, $\mu \in (0,n)$ and $nq + \mu > n$.

Without loss of generality, we may choose a number $m' < n$, close enough to n, and such that $\ell = \frac{n^2}{(n-m')(n+1)}$. Setting $\ell' = \frac{nm'}{n-m'}$, we have

$$\ell < \ell', \quad \frac{n(\ell-1)}{\ell} = \frac{m'(\ell'-1)}{\ell'}$$

and therefore (4.12) becomes

$$|b(x,z,\xi)| \le \Lambda\left(\psi(x) + |z|^{\ell'-1} + |\xi|^{\frac{m'(\ell'-1)}{\ell'}}\right) \tag{4.14}$$

for $|z| \ge 1$ and $|\xi| \ge 1$, while (4.13) takes on the form

$$\begin{aligned}\mathbf{a}(x,z,\xi)\cdot\xi &\ge \gamma|\xi|^n - \Lambda|z|^\ell - \Lambda\varphi(x)^{\frac{n}{n-1}}\\ &\ge \gamma|\xi|^{m'} - \Lambda|z|^{\ell'} - \Lambda\varphi(x)^{\frac{m'}{m'-1}}\end{aligned} \tag{4.15}$$

when $|z| \ge 1$ and $|\xi| \ge 1$ and where, without loss of generality, we have supposed $\varphi(x) \ge 1$.

Defined now the measure

$$d\mathcal{M}' = \left(\chi(x) + \varphi(x)^{\frac{m'}{m'-1}} + \psi(x) + |u(x)|^{\frac{m'^2}{n-m'}}\right)\, dx,$$

we may increase, if necessary, the value of m', maintaining it anyway less than n, in order to have $\frac{m'}{m'-1} < p$, $m' > \frac{n-\lambda}{p} + 1$, $m' > \frac{n-\mu}{q}$ and therefore

$$\mathcal{M}'(B_\rho) \leq K\rho^{n-m'+\varepsilon_0}$$

as above, with a suitable $\varepsilon_0 > 0$.

Considering the function $v(x)$ and the sets Ω_k as defined before, it is immediate that

$$\int_{\{x\in\Omega_k:\ |Dv(x)|<1\}} |Dv(x)|^{m'}\, dx \leq |\Omega_k| \leq k^{m'}\mathcal{M}'(\Omega_k),$$

while

$$\int_{\{x\in\Omega_k:\ |Dv(x)|\geq 1\}} |Dv(x)|^{m'}\, dx$$

can be estimated with the aid of (4.14) and (4.15) as already did when $m < n$. That leads to the bound (4.10) with m' instead of m and it remains to run the same procedure employed above in order to complete the proof of Theorem 2.1. $\square$

4.2. Sharpness of the Hypotheses. We will show, on the level of simple examples built on the m-Laplace operator, that the restrictions on the growths with respect to u and Du and on the Sobolev–Morrey exponents as asked in (2.2) and (2.4) are *sharp* in order to have essential boundedness of the weak solutions to (1.1).

Example 4.1 (The $|u|$-growth $\ell-1$ of $b(x,u,Du)$ is *optimal* for the boundedness.)**.** Let $\varkappa > \ell - 1 > m - 1$. The function

$$u(x) := |x|^{\frac{m}{m-\varkappa-1}} \in W^{1,m}$$

is a local weak solution of the equation

$$\operatorname{div}\left(|Du|^{m-2}Du\right) = C(n,m,\varkappa)|u|^{\varkappa}$$

in the unit ball $B_1(0)$, but $u \notin L^\infty(B_1)$.

Example 4.2 (The gradient growth $\frac{m(\ell-1)}{\ell}$ of $b(x,u,Du)$ is *optimal* for the boundedness.)**.** Let $m < n$ and $\varkappa \in \left(\frac{m(\ell-1)}{\ell}, m\right)$. The function

$$u(x) := |x|^{\frac{m-\varkappa}{m-\varkappa-1}} - 1$$

is a $W^{1,m}_0(B_1)$ weak solution of the Dirichlet problem for the equation

$$\operatorname{div}\left(|Du|^{m-2}Du\right) = C(n,m,\varkappa)|Du|^{\varkappa},$$

but $u \notin L^\infty(B_1)$.

Example 4.3 (The requirements $\varphi \in L^{p,\lambda}(\Omega)$ with $(m-1)p+\lambda > n$ and $\psi \in L^{q,\mu}(\Omega)$ with $mq+\mu > n$ are *sharp* for the boundedness.)**.** Let $B_R = \{x \in \mathbb{R}^n:\ |x| < R < 1\}$ and consider the functions

$$\varphi(x) := \frac{x}{|x|^m \big|\log|x|\big|^{m-1}}$$

and

$$\psi(x) := \frac{(m-n)\log|x| - m + 1}{|x|^m|\log|x||^m}.$$

It is immediate to check that $\varphi \in L^{\frac{n}{m-1}}(B_R;\mathbb{R}^n) \subset L^{p',n-(m-1)p'}\ \forall p' \in \left(1, \frac{n}{m-1}\right]$ but $\varphi \notin L^{p',n-(m-1)p'+\varepsilon}(B_R;\mathbb{R}^n)\ \forall\varepsilon > 0$; and $\psi \in L^{\frac{n}{m}}(B_R) \subset L^{q',n-mq'}(B_R)$ $\forall q' \in \left(1, \frac{n}{m}\right]$, but $\psi \notin L^{q',n-mq'+\varepsilon}(B_R)\ \forall\varepsilon > 0$.

The *unbounded* function

$$u(x) = \log\left(\frac{\log|x|}{\log R}\right)$$

is a $W_0^{1,m}(B_R)$-weak solution to the homogeneous Dirichlet problems of both

$$\operatorname{div}\left(|Du|^{m-2}Du - \varphi(x)\right) = 0$$

and

$$\operatorname{div}\left(|Du|^{m-2}Du\right) = \psi(x).$$

5. Global Hölder Continuity

Let us start with the Hölder regularity of the weak solutions in the *interior* of Ω as claimed in Corollary 2.2.

Proof of Corollary 2.2. Let $m = n$. Then Lemma 3.8 implies $u \in W^{1,m_0}(\Omega)$ with $m_0 > n$ and thus the interior Hölder continuity of u with exponent $1 - \frac{n}{m_0}$ follows from the Morrey lemma.

Suppose therefore $m < n$. We have then $\ell = \frac{nm}{n-m}$ (cf. (2.3)) and, taking into account the essential boundedness of u given by Theorem 2.1, the structure conditions (2.2) and (2.4) can be rewritten as

$$\begin{aligned}
|\mathbf{a}(x,z,\xi)| &\le \Lambda\left(\varphi'(x) + |\xi|^{m-1}\right),\\
|b(x,z,\xi)| &\le \Lambda\left(\psi'(x) + |\xi|^{\frac{m(\ell-1)}{\ell}}\right) \le \Lambda\left(\psi'(x) + |\xi|^{m}\right),\\
\mathbf{a}(x,z,\xi)\cdot\xi &\ge \gamma|\xi|^m - \varphi''(x)
\end{aligned}$$

for a.a. $x \in \Omega$ and all $(z,\xi) \in \mathbb{R}\times\mathbb{R}^n$, where

$$\varphi'(x) = \varphi(x) + M^{\frac{n(m-1)}{n-m}}, \qquad \psi'(x) = \psi(x) + M^{\frac{nm-n+m}{n-m}} + 1,$$

$$\varphi''(x) = \Lambda\left(M^{\frac{nm}{n-m}} + \varphi(x)^{\frac{m}{m-1}}\right).$$

Straightforward calculations, based on the hypotheses $\varphi \in L^{p,\lambda}(\Omega)$, $p > \frac{m}{m-1}$, $(m-1)p + \lambda > n$ and $\psi \in L^{q,\mu}(\Omega)$, $q > \frac{mn}{mn+m-n}$, $mq + \mu > n$, give

$$\int_{B_\rho} \varphi'(x)\,dx \le C\left(n,m,\|\varphi\|_{L^{p,\lambda}(\Omega)},M,\operatorname{diam}\Omega\right)\rho^{n-\frac{n-\lambda}{p}} = C\rho^{n-m+1+\varepsilon_1}$$

with $\varepsilon_1 = m - 1 - \frac{n-\lambda}{p} > 0$,

$$\int_{B_\rho} \psi'(x)\,dx \le C\left(n,m,\|\psi\|_{L^{q,\mu}(\Omega)},M,\operatorname{diam}\Omega\right)\rho^{n-\frac{n-\mu}{q}} = C\rho^{n-m+\varepsilon_2}$$

with $\varepsilon_2 = m - \frac{n-\mu}{q} > 0$, and

$$\int_{B_\rho} \varphi''(x)\,dx \le C\left(n,m,\Lambda,\|\varphi\|_{L^{p,\lambda}(\Omega)},M,\operatorname{diam}\Omega\right)\rho^{n-\frac{m(n-\lambda)}{p(m-1)}} = C\rho^{n-m+\varepsilon_3}$$

with $\varepsilon_3 = m - \frac{m(n-\lambda)}{p(m-1)} > 0$.

At this point, the claim of Corollary 2.2 follows from the Harnack inequality proved by Lieberman (see [17, Theorem 4.1] and [26, Theorem 2.2]) and standard covering arguments. $\square$

To proceed further with the more delicate question of Hölder continuity *up to the boundary* of Ω, we need the following result ensuring suitable growth estimate for the gradient over small balls.

Lemma 5.1. *Assume* (2.2) *and* (2.4)*, and let u be a weak solution to the problem* (1.1) *extended az zero outside Ω.*

Let B_ρ be a ball of radius $\rho \in (0, \operatorname{diam}\Omega)$ and centered at a point of $\partial\Omega$, and $\eta \in C_0^\infty(B_{\rho/2})$ with $|D\eta| \le c/\rho$. There exists a constant C depending on the same quantities as M in (2.6)*, such that*

$$\int_{B_{\rho/2}} |D(\eta w^{-1})|^m \, dx \le C\big(M(\rho) + A(\rho)\big)\left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\right)^{m-1} \rho^{n-m}$$

where $M(\rho) = \operatorname{ess\,sup}_{B_\rho} \overline{u}$, $\overline{u} = \max\{u, 0\}$, $A(\rho) = \rho + \|\varphi\|_{L^{p,\lambda}(B_\rho)}^{\frac{1}{m-1}} + \|\psi\|_{L^{q,\mu}(B_\rho)}^{\frac{1}{m}}$ and $w^{-1} = M(\rho) + A(\rho) - \overline{u}$.

Proof. Let $1 \le \theta < \min\left\{\frac{p(m-1)}{m}, q\right\}$ be arbitrary and consider the measure

$$d\mathcal{M}_\theta := \left(\varphi(x)^{\frac{\theta m}{m-1}} + \psi(x)^\theta\right)\, dx$$

where, as before, φ and ψ are supposed to be extended as zero outside Ω.

Given a ball B_ρ of radius ρ, we have

$$\int_{B_\rho} \varphi(x)^{\frac{\theta m}{m-1}}\, dx \le \|\varphi\|_{L^{p,\lambda}(B_\rho)}^{\frac{\theta m}{m-1}} \rho^{n - \frac{\theta m(n-\lambda)}{p(m-1)}} \le A(\rho)^{\theta m} \rho^{n - \frac{\theta m(n-\lambda)}{p(m-1)}}$$
$$\int_{B_\rho} \psi(x)^\theta\, dx \le \|\psi\|_{L^{q,\mu}(B_\rho)}^\theta \rho^{n - \frac{\theta(n-\mu)}{q}} \le A(\rho)^{\theta m} \rho^{n - \frac{\theta(n-\mu)}{q}}.$$

Therefore, setting

$$\alpha_\theta := \min\left\{n - \frac{\theta m(n-\lambda)}{p(m-1)}, n - \frac{\theta(n-\mu)}{q}\right\} > 0, \tag{5.1}$$

we get

$$\mathcal{M}_\theta(B_\rho) \le C A(\rho)^{\theta m} \rho^{\alpha_\theta}$$

with a constant C depending on known quantities.

Now, let B_ρ be a ball of radius ρ centered at a boundary point and use $v := \eta^m e^{\frac{\Lambda}{\gamma}\overline{u}} v_0$ as test function in (2.5), where $\eta \in C_0^\infty(B_\rho)$, $v_0 = w^\beta - (M(\rho) + A(\rho))^{-\beta}$ and $\beta > 0$ is a parameter under control. Having in mind that $|D\overline{u}| = |Du|$ a.e. $\{x \in \Omega\colon u(x) > 0\}$, we obtain

$$\int \eta^m e^{\frac{\Lambda}{\gamma}\overline{u}} \left(\beta w^{\beta+1} + \frac{\Lambda}{\gamma} v_0\right) \mathbf{a}(x, u(x), Du(x)) \cdot Du\, dx$$
$$+ \int m\eta^{m-1} e^{\frac{\Lambda}{\gamma}\overline{u}} v_0\, \mathbf{a}(x, u(x), Du(x)) \cdot D\eta\, dx + \int b(x, u(x), Du(x)) v\, dx = 0$$

and hereafter all the integrals will be taken over $B_\rho \cap \{x \in \Omega\colon u(x) > 0\}$. Using that

$$|Du|^{\frac{m(\ell-1)}{\ell}} \le |Du|^m + 1$$

in view of the Young inequality, we get

$$\begin{aligned}\gamma\beta\int\eta^m e^{\frac{\Lambda}{\gamma}\overline{u}}w^{\beta+1}|Du|^m\,dx \le \Lambda m\int\eta^{m-1}e^{\frac{\Lambda}{\gamma}\overline{u}}v_0\Big(|Du|^{m-1}+|u|^{\frac{\ell(m-1)}{m}}+\varphi\Big)|D\eta|\,dx\\ +\Lambda\int\eta^m e^{\frac{\Lambda}{\gamma}\overline{u}}v_0\left(|u|^{\ell-1}+\psi+1\right)\,dx\\ +\Lambda\int\eta^m e^{\frac{\Lambda}{\gamma}\overline{u}}\left(\beta w^{\beta+1}+\frac{\Lambda}{\gamma}v_0\right)\left(|u|^{\ell}+\varphi^{\frac{m}{m-1}}\right)\,dx\end{aligned}$$

as consequence of (2.2) and (2.4). Since $v_0\le w^\beta$ and

$$w^{-1}\le C\left(\|u\|_{L^\infty(\Omega)}+\|\varphi\|_{L^{p,\lambda}(\Omega)}^{\frac{1}{m-1}}+\|\psi\|_{L^{q,\mu}(\Omega)}^{\frac{1}{m}}+\operatorname{diam}\Omega\right),$$

it follows

$$\begin{aligned}\beta\int\eta^m w^{\beta+1}|Du|^m\,dx \le C\int\eta^{m-1}w^\beta|Du|^{m-1}|D\eta|\,dx\\ +C\int\eta^{m-1}w^\beta\left(\varphi+1\right)|D\eta|\,dx\\ +C\int\eta^m w^{\beta+1}\left(\psi+1\right)\,dx\\ +C(1+\beta)\int\eta^m w^{\beta+1}\left(\varphi^{\frac{m}{m-1}}+1\right)\,dx\end{aligned}\tag{5.2}$$

where the constant C depends on known quantities and on $\|Du\|_{L^m(\Omega)}$ through M in (2.6). We apply the Young inequality to get

$$\begin{aligned}\int\eta^{m-1}w^\beta|Du|^{m-1}|D\eta|\,dx \le \varepsilon\int\eta^m w^{\beta+1}|Du|^m\,dx\\ +C\varepsilon^{1-m}\int w^{\beta-m+1}|D\eta|^m\,dx\\ \int\eta^{m-1}w^\beta\left(\varphi+1\right)|D\eta|\,dx \le \beta\int\eta^m w^{\beta+1}\left(\varphi^{\frac{m}{m-1}}+1\right)\,dx\\ +C\beta^{1-m}\int w^{\beta-m+1}|D\eta|^m\,dx\end{aligned}$$

for any $\varepsilon>0$. Choosing $\varepsilon=\frac{\beta}{2C}$ with appropriate C above, we obtain from (5.2)

$$\begin{aligned}\beta\int\eta^m w^{\beta+1}|Du|^m\,dx \le C(1+\beta)\int\eta^m w^{\beta+1}\,dx\\ +C(1+\beta)\int\eta^m w^{\beta+1}\left(\varphi^{\frac{m}{m-1}}+\psi\right)\,dx\\ +C\beta^{1-m}\int w^{\beta-m+1}|D\eta|^m\,dx.\end{aligned}\tag{5.3}$$

Take now $\beta=m-1$ in (5.3). We have $M(\rho)-\overline{u}\ge0$ whence $w\le A(\rho)^{-1}$ and the Poincaré inequality yields

$$\int\eta^m w^m\,dx\le A(\rho)^{-m}\int\eta^m\,dx\le C\int|D\eta|^m\,dx.$$

Further on,

$$\int \eta^m w^m \left(\varphi^{\frac{m}{m-1}} + \psi\right)\, dx = \int \eta^m w^m d\mathcal{M}_1 \ \leq A(\rho)^{-m} \int \eta^m d\mathcal{M}_1.$$

To estimate the term on the right-hand side above, we will distinguish between the cases $m < n$ and $m = n$. Thus, if $m < n$ we apply the Adams trace inequality from Proposition 3.3 with

$$\alpha_0 = \alpha_1, \qquad s = \frac{\alpha_1 m}{n-m}, \qquad r = m,$$

and where α_1 is taken from (5.1) with $\theta = 1$. We have

$$\begin{aligned}
A(\rho)^{-m} \int \eta^m d\mathcal{M}_1 &\leq A(\rho)^{-m} \left(\int \eta^s d\mathcal{M}_1 \right)^{m/s} \left(\int d\mathcal{M}_1 \right)^{1-m/s} \\
&\leq A(\rho)^{-m} \left(CA(\rho)^{\frac{m^2}{s}} \int |D\eta|^m\, dx \right) \left(CA(\rho)^m \rho^{\alpha_1} \right)^{1-m/s} \\
&\leq C\rho^{\alpha_1 - n + m} \int |D\eta|^m\, dx \\
&\leq C \int |D\eta|^m\, dx,
\end{aligned}$$

where $\alpha_1 - n + m > 0$ and $0 < \rho < \operatorname{diam} \Omega$ have been used in the last bound.

If instead $m = n$, we employ once again Proposition 3.3, but

$$\alpha_0 = \alpha_1, \qquad s = n, \qquad r = \frac{n^2}{\alpha_1 + n}$$

now. Thus

$$\begin{aligned}
A(\rho)^{-n} \int \eta^n d\mathcal{M}_1 &\leq C \left(\int |D\eta|^r\, dx \right)^{n/r} \leq C \left(\int |D\eta|^n\, dx \right) \left(\int dx \right)^{n/r-1} \\
&\leq C\rho^{\alpha_1} \int |D\eta|^n\, dx \\
&\leq C \int |D\eta|^n\, dx
\end{aligned}$$

thanks to $\alpha_1 > 0$ and $0 < \rho < \operatorname{diam} \Omega$.

This way,

$$\int \eta^m w^m \left(\varphi^{\frac{m}{m-1}} + \psi\right)\, dx \leq C \int |D\eta|^m\, dx$$

and (5.3) with $\beta = m - 1$ becomes

$$\int \eta^m |D(\log w)|^m\, dx \leq C \int |D\eta|^m\, dx$$

for each $0 \leq \eta \in C_0^\infty(B_\rho)$.

Choosing appropriately η, we are in a position to apply Proposition 3.5 that asserts existence of constants C and $\sigma_0 >$ such that

$$\int_{B_{3\rho/4}} w^{-\sigma}\, dx \int_{B_{3\rho/4}} w^{\sigma}\, dx \leq C\rho^{2n} \qquad \text{for all } |\sigma| \leq \sigma_0. \tag{5.4}$$

Consider now the cases $\beta \neq m-1$ in (5.3). For, we multiply the both sides of (5.3) by β^{m-1} which rewrites it as

$$\beta^m \int \eta^m w^{\beta+1}|Du|^m\, dx \leq C\int w^{\beta-m+1}|D\eta|^m\, dx + C(1+\beta^m)\int \eta^m w^{\beta+1}\left(\varphi^{\frac{m}{m-1}}+\psi+1\right)\, dx. \tag{5.5}$$

Setting $\beta = mt+m-1>0$, we have

$$|D(\eta w^t)|^m \leq 2^{m-1}\left(w^{mt}|D\eta|^m + |t|^m\eta^m w^{mt+m}|Du|^m\right)$$

and the use of (5.5) with $\beta = mt+m-1$ gives

$$\int |D(\eta w^t)|^m\, dx \leq\ C\left(1+\frac{|t|^m}{(mt+m-1)^m}\right)\int w^{mt}|D\eta|^m\, dx + C\left(|t|^m+\frac{|t|^m}{(mt+m-1)^m}\right)\int \eta^m w^{mt+m}\left(\varphi^{\frac{m}{m-1}}+\psi+1\right)\, dx.$$

We have

$$\frac{t}{mt+m-1} < \frac{1}{m} \quad \forall t>0,$$

while $|t|^m < \left(\frac{m-1}{m}\right)^m$ and $\frac{|t|^m}{(mt+m-1)^m}$ is a positive and decreasing function whenever $t \in \left(\frac{1-m}{m},0\right)$.

Thus, defining

$$N(t) := \begin{cases} 1+t^m & \text{if } t>0,\\ 1+\dfrac{|t|^m}{(mt+m-1)^m} & \text{if } \dfrac{1-m}{m}<t\leq 0,\end{cases}$$

the last bound takes on the form

$$\int |D(\eta w^t)|^m\, dx \leq CN(t)\left(\int w^{mt}|D\eta|^m\, dx + \int \eta^m w^{mt+m}\left(\varphi^{\frac{m}{m-1}}+\psi+1\right)\, dx\right). \tag{5.6}$$

The Young inequality and the definition of the measure $\mathcal{M}_\theta$ yield

$$\begin{aligned}\int \eta^m w^{mt+m}\left(\varphi^{\frac{m}{m-1}}+\psi\right)\, dx &\leq \varepsilon\int \eta^m w^{mt+m}\left(\varphi^{\frac{\theta m}{m-1}}+\psi^\theta\right)\, dx\\ &\quad + C\varepsilon^{\frac{1}{1-\theta}}\int \eta^m w^{mt+m}\, dx\\ &= \varepsilon\int \eta^m w^{mt+m} d\mathcal{M}_\theta + C\varepsilon^{\frac{1}{1-\theta}}\int \eta^m w^{mt+m}\, dx,\end{aligned} \tag{5.7}$$

where θ is such that $1<\theta<\min\left\{\frac{p(m-1)}{n-\lambda}, \frac{mq}{n-\mu}\right\}$. In order to estimate the first term on the right-hand side of (5.7) we will employ once again Proposition 3.3 distinguishing between the cases $m<n$ and $m=n$.

Let $m<n$. Taking

$$\alpha_0 = \alpha_\theta, \qquad s = \frac{\alpha_\theta m}{n-m}, \qquad r = m$$

in Proposition 3.3 and remembering $w \le A(\rho)^{-1}$, we get

$$\begin{aligned}\int \eta^m w^{mt+m}\, d\mathcal{M}_\theta &\le A(\rho)^{-m} \int \left(\eta w^t\right)^m d\mathcal{M}_\theta \\ &\le A(\rho)^{-m} \left(\int \left(\eta w^t\right)^s d\mathcal{M}_\theta\right)^{m/s} \left(\int\, d\mathcal{M}_\theta\right)^{1-m/s} \\ &\le CA(\rho)^{(\theta-1)m+\alpha_\theta-(n-m)} \int |D(\eta w^t)|^m\, dx \\ &\le C \int |D(\eta w^t)|^m\, dx\end{aligned}$$

since $\theta > 1$ and $\alpha_\theta > n-m$.

In case $m = n$, the application of the Adams trace inequality with

$$\alpha_0 = \alpha_\theta, \qquad s = n, \qquad r = \frac{n^2}{\alpha_\theta + n}$$

gives

$$\begin{aligned}\int \eta^n w^{nt+n} d\mathcal{M}_\theta &\le A(\rho)^{-n} \int \left(\eta w^t\right)^n d\mathcal{M}_\theta \\ &\le CA(\rho)^{(\theta-1)n} \left(\int |D(\eta w^t)|^r\, dx\right)^{n/r} \\ &\le CA(\rho)^{(\theta-1)n} \left(\int |D(\eta w^t)|^n\, dx\right) \left(\int\, dx\right)^{n/r-1} \\ &= CA(\rho)^{(\theta-1)n+\alpha_\theta} \int |D(\eta w^t)|^n\, dx \\ &\le C \int |D(\eta w^t)|^n\, dx.\end{aligned}$$

Therefore

$$\int \eta^m w^{mt+m}\, d\mathcal{M}_\theta \le C \int |D(\eta w^t)|^m\, dx$$

and (5.7) becomes

$$\int \eta^m w^{mt+m} \left(\varphi^{\frac{m}{m-1}} + \psi\right)\, dx \le C\varepsilon \int |D(\eta w^t)|^m\, dx + C\varepsilon^{\frac{1}{1-\theta}} \int \eta^m w^{mt+m}\, dx$$

with $\varepsilon > 0$ under control. In particular, choosing $\varepsilon = \frac{1}{2CN(t)}$ above and having in mind $N(t) \ge 1$, we get from (5.6)

$$\int |D(\eta w^t)|^m\, dx \le CK(t) \int w^{mt} \left(\eta^m w^m + |D\eta|^m\right)\, dx \tag{5.8}$$

with

$$K(t) := \left(N(t)\right)^{\frac{\theta}{\theta-1}}.$$

Let us take now a cut-off function $\eta \in C_0^\infty(B_r)$ such that $0 \le \eta \le 1$, $\eta = 1$ on B_s and $|D\eta| \le \frac{c}{r-s}$ where $0 < s < r \le \rho$. Employing the Sobolev inequality, we get

$$\begin{aligned}
(5.9) \qquad \left(\int_{B_s} \left(w^t\right)^\ell \, dx\right)^{m/\ell} &= \left(\int_{B_s} \left(\eta w^t\right)^\ell \, dx\right)^{m/\ell} \\
&\le C\left(\int_{B_r} |D(\eta w^t)|^{\frac{\ell n}{\ell+n}} \, dx\right)^{m(\ell+n)/(\ell n)} \\
&\le C\left(\int_{B_r} |D(\eta w^t)|^m \, dx\right)\left(\int_{B_r} dx\right)^{m(\ell+n)/(\ell n)-1} \\
&\le \frac{CK(t)}{(r-s)^m}\rho^{\frac{m(\ell+n)-\ell n}{\ell}} \int_{B_r} w^{mt} \, dx
\end{aligned}$$

from (5.8) since $\eta w \le \frac{1}{\rho} \le \frac{1}{r-s}$.

Let $\rho_k = \rho\left(\frac{1}{2} + \frac{1}{2^{k+2}}\right)$ for $k = 0, 1, \ldots$ and let $t_0 > 0$ be any number such that $mt_0 \le \sigma_0$ with σ_0 appearing in (5.4). Making use of the simple inequalities $e^{2mt} \ge (1+t)^{2m} \ge 1 + t^{2m}$ valid for all $t \ge 0$ and all $m \ge \frac{1}{2}$, and remembering the properties of the function $K(t)$, we have

$$K(t) = (1+t^m)^{\frac{\theta}{\theta-1}} \le e^{\frac{2m\theta}{\theta-1}\sqrt{t}} \quad \forall t > 0.$$

Thus, taking $t = t_0\left(\frac{\ell}{m}\right)^k > 0$ and using (5.9) with $s = \rho_{k+1}$ and $r = \rho_k$, we get

$$\begin{aligned}
&\left(\int_{B_{\rho_{k+1}}} w^{mt_0\left(\frac{\ell}{m}\right)^{k+1}} \, dx\right)^{(m/\ell)^{k+1}} \\
&\qquad \le \left(2^k C\right)^{m\left(\frac{m}{\ell}\right)^k} \rho^{\frac{n(m-\ell)}{\ell}\left(\frac{m}{\ell}\right)^k} e^{\frac{2m\theta}{\theta-1}\sqrt{t_0}\left(\frac{m}{\ell}\right)^{\frac{k}{2}}} \left(\int_{B_{\rho_k}} w^{mt_0\left(\frac{\ell}{m}\right)^k} \, dx\right)^{(m/\ell)^k}
\end{aligned}$$

for $k = 0, 1, \ldots$. Iteration of these inequalities from 0 to $N \in \mathbb{N}$ yields

$$\begin{aligned}
&\left(\int_{B_{\rho_{N+1}}} w^{mt_0\left(\frac{\ell}{m}\right)^{N+1}} \, dx\right)^{(m/\ell)^{N+1}} \\
&\quad \le 2^{m\sum_{k=0}^N k\left(\frac{m}{\ell}\right)^k} C^{m\sum_{k=0}^N\left(\frac{m}{\ell}\right)^k} e^{\frac{2m\theta}{\theta-1}\sqrt{t_0}\sum_{k=0}^N\left(\frac{m}{\ell}\right)^{\frac{k}{2}}} \rho^{\frac{n(m-\ell)}{\ell}\sum_{k=0}^N\left(\frac{m}{\ell}\right)^k} \int_{B_{\rho_0}} w^{mt_0} \, dx
\end{aligned}$$

and passage to the limit as $N \to +\infty$ gives

$$(5.10) \qquad \operatorname*{ess\,sup}_{B_{\rho/2}} w^{mt_0} = \left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\right)^{-mt_0} \le C\rho^{-n}\int_{B_{3\rho/4}} w^{mt_0} \, dx.$$

This way, it follows from (5.4) and (5.10) that

$$(5.11) \qquad \rho^{-n}\int_{B_{3\rho/4}} w^{-mt_0} \, dx \le C\left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\right)^{mt_0}$$

for any $t_0 > 0$ such that $0 < mt_0 \le \sigma_0$.

To proceed further, we set $\sigma = -mt_1$ where $t_1 = t_1(m,n) < 0$ will be chosen in the sequel. Let $0 < \sigma < m-1$ and define $\sigma_1 = \sigma\left(\frac{m}{\ell}\right)^\kappa$ where κ is a positive integer

for which $m-1 \le \sigma_0 \left(\frac{\ell}{m}\right)^{\kappa}$ and σ_0 is taken from (5.4). Obviously, $0 < \sigma_1 \left(\frac{\ell}{m}\right)^k \le \sigma$ for $0 \le k \le \kappa$. Thus $\frac{1-m}{m} < -\frac{\sigma}{m} \le -\frac{\sigma_1}{m}\left(\frac{\ell}{m}\right)^k$ and therefore

$$K\left(-\frac{\sigma_1}{m}\left(\frac{\ell}{m}\right)^k\right) \le K\left(-\frac{\sigma}{m}\right)$$

for $0 \le k \le \kappa$ since K is a decreasing function on $\left(\frac{1-m}{m}, 0\right)$.

We take now $\rho_k = \frac{\rho}{4}\left(3 - \frac{k}{\kappa+1}\right)$ for $0 \le k \le \kappa+1$ and apply (5.9) with $s = \rho_{k+1}$, $r = \rho_k$ and $t = -\frac{\sigma_1}{m}\left(\frac{\ell}{m}\right)^k$ in order to get

$$\begin{aligned}&\left(\int_{B_{\rho_{k+1}}} w^{-\sigma_1\left(\frac{\ell}{m}\right)^{k+1}}\, dx\right)^{(m/\ell)^{k+1}}\\ &\qquad \le \left(CK\left(-\frac{\sigma}{m}\right)4^m(\kappa+1)^m \rho^{\frac{n(m-\ell)}{\ell}}\right)^{\left(\frac{m}{\ell}\right)^k}\left(\int_{B_{\rho_k}} w^{-\sigma_1\left(\frac{\ell}{m}\right)^k}\, dx\right)^{(m/\ell)^k}\end{aligned}$$

for $0 \le k \le \kappa$. Iteration of these inequalities for $0 \le k \le \kappa$ gives

$$\begin{aligned}&\left(\int_{B_{\rho/2}} w^{-\sigma_1\left(\frac{\ell}{m}\right)^{\kappa+1}}\, dx\right)^{(m/\ell)^{\kappa+1}}\\ &\qquad \le \left(CK\left(-\frac{\sigma}{m}\right)4^m(\kappa+1)^m\right)^{\sum_{k=0}^{\kappa}\left(\frac{m}{\ell}\right)^k} \rho^{\frac{n(m-\ell)}{\ell}\sum_{k=0}^{\kappa}\left(\frac{m}{\ell}\right)^k}\int_{B_{3\rho/4}} w^{-\sigma_1}\, dx,\end{aligned}$$

whence

$$\rho^{-n}\int_{B_{\rho/2}} w^{-\sigma_1\left(\frac{\ell}{m}\right)^{\kappa+1}}\, dx \le C\left(\rho^{-n}\int_{B_{3\rho/4}} w^{-\sigma_1}\, dx\right)^{(\ell/m)^{\kappa+1}}.$$

Remembering $0 < \sigma_1 < \sigma_0$, we obtain from (5.11) with $\sigma_1 = mt_0$ that

$$\rho^{-n}\int_{B_{\rho/2}} w^{t_1\ell}\, dx \le C\left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\right)^{-t_1\ell} \tag{5.12}$$

for each $t_1 < 0$ such that $0 < -mt_1 < m-1$.

Take now $v = \eta^m e^{\frac{\Lambda}{\gamma}\overline{u}}\overline{u}$ as a test function in (2.5). Keeping in mind (2.2), (2.4) and (2.6), we obtain

$$\begin{aligned}\int \eta^m |Du|^m\, dx \le{}& CM(\rho)\int \eta^{m-1}|Du|^{m-1}|D\eta|\, dx\\ &+ CM(\rho)\int \eta^{m-1}|D\eta|(1+\varphi)\, dx\\ &+ C\int \eta^m\left(\varphi^{\frac{m}{m-1}} + \psi\right)\, dx + C\int \eta^m\, dx.\end{aligned} \tag{5.13}$$

Fix $\eta \in C_0^\infty\left(B_{\rho/2}\right)$ such that $0 \le \eta \le 1$ and $|D\eta| \le c/\rho$. To estimate the terms on the right-hand side in (5.13), we take a $t_2 < 0$ such that $1 < (1+t_2)m < \frac{\ell}{m}$.

Thus

$$
\begin{aligned}
\int \eta^{m-1}|Du|^{m-1}|D\eta|\,dx &= \int \big(\eta w^{1+t_2}|Du|\big)^{m-1}\big(w^{-(1+t_2)(m-1)}|D\eta|\big)\,dx\\
&= C\int \big(\eta|D(w^{t_2})|\big)^{m-1}\big(w^{-(1+t_2)(m-1)}|D\eta|\big)\,dx\\
&\le C\left(\int \big(\eta|D(w^{t_2})|\big)^{m}\,dx\right)^{1-1/m}\left(\int \big(w^{-(1+t_2)(m-1)}|D\eta|\big)^{m}\,dx\right)^{1/m}.
\end{aligned}
$$

The two terms above will be estimated with the aid of (5.8) and (5.12), respectively. Precisely, applying (5.8) with $t = t_2$, we have

$$
\begin{aligned}
\left(\int \big(\eta|D(w^{t_2})|\big)^{m}\,dx\right)^{1-1/m} &\le C\left(\int \big(|D(\eta w^{t_2})|\big)^{m}\,dx + \int \big(w^{t_2}|D\eta|\big)^{m}\,dx\right)^{1-1/m}\\
&\le C\left(\Big(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\Big)^{-mt_2}\rho^{n-m}\right)^{1-1/m}
\end{aligned}
$$

since $w \le \big(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\big)^{-1}$ on $B_{\rho/2}$ as well as $w \le A(\rho)^{-1} \le \rho^{-1}$.

In the same manner, (5.12) with $t_1 = -\frac{(1+t_2)(m-1)m}{\ell}$ leads to

$$
\begin{aligned}
&\left(\int \big(w^{-(1+t_2)(m-1)}|D\eta|\big)^{m}\,dx\right)^{1/m}\\
&\qquad \le C\left(\Big(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\Big)^{(1+t_2)(m-1)m}\rho^{n-m}\right)^{1/m}
\end{aligned}
$$

whence

$$
\int \eta^{m-1}|Du|^{m-1}|D\eta|\,dx \le C\Big(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\Big)^{m-1}\rho^{n-m}.
$$

Further on,

$$
\begin{aligned}
\int \eta^{m-1}|D\eta|(1+\varphi)\,dx &\le C\left(\int \eta^m\left(1+\varphi^{\frac{m}{m-1}}\right)\,dx\right)^{1-1/m}\left(\int |D\eta|^m\,dx\right)^{1/m}\\
&\le C\left(\rho^n + \mathcal{M}_1(B_{\rho/2})\right)^{\frac{m-1}{m}}\rho^{\frac{n-m}{m}}\\
&\le C\left(\rho^m\rho^{n-m} + A(\rho)^m\rho^{\alpha_1}\right)^{\frac{m-1}{m}}\rho^{\frac{n-m}{m}}\\
&\le C\left(A(\rho)^m\rho^{n-m}\right)^{\frac{m-1}{m}}\rho^{\frac{n-m}{m}}\\
&= CA(\rho)^{m-1}\rho^{n-m}
\end{aligned}
$$

because of $\mathcal{M}_1(B_{\rho/2}) \le A(\rho)^m\rho^{\alpha_1}$ (cf. (5.1)), $\rho \le A(\rho)$ and $\alpha_1 > n-m$.

Similarly,

$$
\int \eta^m\left(\varphi^{\frac{m}{m-1}} + \psi\right)\,dx = \int \eta^m\,d\mathcal{M}_1 \le CA(\rho)^m\rho^{n-m}
$$

and

$$
\int \eta^m\,dx = C\rho^m\int |D\eta|^m\,dx \le CA(\rho)^m\rho^{n-m}
$$

in view of the Poincaré inequality.

Therefore, (5.13) yields

$$
\int \eta^m|Du|^m\,dx \le C\big(M(\rho)+A(\rho)\big)\Big(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\Big)^{m-1}\rho^{n-m}. \tag{5.14}
$$

On the other hand,

$$\int |D(\eta w^{-1})|^m \le C \left(\int w^{-m}|D\eta|^m \, dx + \int \eta^m |Du|^m \, dx \right)$$

and, keeping in mind $w^{-1} \le M(\rho) + A(\rho)$, we apply (5.12) with $t_1 = \frac{1-m}{\ell}$ to get

$$\begin{aligned}\int w^{-m}|D\eta|^m \, dx &\le C\rho^{-m} \int w^{-1}w^{1-m} \, dx \\ &\le C\big(M(\rho)+A(\rho)\big)\rho^{-m} \int w^{1-m} \, dx \\ &\le C\big(M(\rho)+A(\rho)\big)\left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\right)^{m-1} \rho^{n-m}.\end{aligned}$$

The last bound, together with (5.14) gives

$$\int |D(\eta w^{-1})|^m \, dx \le C\big(M(\rho)+A(\rho)\big)\left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\right)^{m-1} \rho^{n-m}$$

and this completes the proof of Lemma 5.1. □

Once having the result of Lemma 5.1 it is easy to extend the Hölder continuity of the weak solutions up to the boundary of Ω, thanks of the P-thickness condition (2.1).

5.1. Proof of Theorem 2.3. Let $x_0 \in \partial\Omega$ be any point and set $B_\rho = B_\rho(x_0)$ for the sake of brevity. Since $(m-1)p + \lambda > n$ and $mq + \mu > n$, there exist positive constants $\overline{\lambda}$ and $\overline{\mu}$ such that $n < (m-1)p + \overline{\lambda} < (m-1)p + \lambda$, $n < mq + \overline{\mu} < mq + \mu$. It follows from Proposition 3.1 that $L^{p,\lambda}(B_\rho) \subset L^{p,\overline{\lambda}}(B_\rho)$ and $L^{q,\mu}(B_\rho) \subset L^{q,\overline{\mu}}(B_\rho)$. In particular, since $\varphi \in L^{p,\lambda}$ and $\psi \in L^{q,\mu}$, we get from the definition

$$\begin{aligned}\|\varphi\|^p_{L^{p,\overline{\lambda}}(B_\rho)} &\le \|\varphi\|^p_{L^{p,\lambda}(B_\rho)} \rho^{\lambda-\overline{\lambda}}, \\ \|\psi\|^q_{L^{q,\overline{\mu}}(B_\rho)} &\le \|\psi\|^q_{L^{q,\mu}(B_\rho)} \rho^{\mu-\overline{\mu}}.\end{aligned} \tag{5.15}$$

In fact, for any $B_s(y) \subset \mathbb{R}^n$, we have

$$\begin{aligned}\frac{1}{s^{\overline{\lambda}}} \int_{B_\rho \cap B_s(y)} \varphi^p \, dx &= s^{\lambda-\overline{\lambda}} \frac{1}{s^\lambda} \int_{B_\rho \cap B_s(y)} \varphi^p \, dx \le \rho^{\lambda-\overline{\lambda}} \frac{1}{s^\lambda} \int_{B_\rho \cap B_s(y)} \varphi^p \, dx \\ &\le \|\varphi\|^p_{L^{p,\lambda}(B_\rho)} \rho^{\lambda-\overline{\lambda}}\end{aligned}$$

if $\rho \ge s$ and

$$\begin{aligned}\frac{1}{s^{\overline{\lambda}}} \int_{B_\rho \cap B_s(y)} \varphi^p \, dx &\le \frac{1}{\rho^{\overline{\lambda}}} \int_{B_\rho \cap B_s(y)} \varphi^p \, dx \le \rho^{\lambda-\overline{\lambda}} \frac{1}{\rho^\lambda} \int_{B_\rho} \varphi^p \, dx \\ &\le \|\varphi\|^p_{L^{p,\lambda}(B_\rho)} \rho^{\lambda-\overline{\lambda}}\end{aligned}$$

if $\rho \le s$.

Take now a cut-off function $\eta \in C_0^\infty\left(B_{\rho/2}\right)$ so that $0 \le \eta \le 1$, $\eta = 1$ on $B_{\rho/4}$ and $|D\eta| \le C/\rho$. Since $\mathbb{R}^n \setminus \Omega$ is P-thick, it is also m-thick whence, according to (2.1),

$$\mathrm{Cap}_m\big(B_{\rho/4} \setminus \Omega, B_{\rho/2}\big) \ge A_\Omega \, \mathrm{Cap}_m\big(B_{\rho/4}, B_{\rho/2}\big) = C\rho^{n-m} \quad \forall \rho \le r_0.$$

On the other hand,

$$B_{\rho/4} \setminus \Omega \subset \left\{x \in B_{\rho/4} \colon \ \overline{u}(x) = 0\right\}$$

and therefore

$$\mathrm{Cap}_m\Big(\left\{x\in B_{\rho/4}:\ \overline{u}(x)=0\right\},B_{\rho/2}\Big)\geq \mathrm{Cap}_m\big(B_{\rho/4}\setminus\Omega,B_{\rho/2}\big).$$

We have $\eta w^{-1}=M(\rho)+\overline{A}(\rho)$ on $\left\{x\in B_{\rho/4}:\ \overline{u}(x)=0\right\}$ with

$$\overline{A}(\rho)=\rho+\|\varphi\|^{\frac{1}{m-1}}_{L^{p,\overline{\lambda}}(B_\rho)}+\|\psi\|^{\frac{1}{m}}_{L^{q,\overline{\mu}}(B_\rho)}$$

and thus

$$\int_{B_{\rho/2}}\left|D\left(\frac{\eta w^{-1}}{M(\rho)+\overline{A}(\rho)}\right)\right|^m\,dx\geq \mathrm{Cap}_m\Big(\left\{x\in B_{\rho/4}:\ \overline{u}(x)=0\right\},B_{\rho/2}\Big).$$

Putting together all these inequalities, Lemma 5.1 gives

$$\begin{aligned}\rho^{n-m}&\leq C\,\mathrm{Cap}_p\Big(\left\{x\in B_{\rho/4}:\ \overline{u}(x)=0\right\},B_{\rho/2}\Big)\\&\leq C\big((M(\rho)+\overline{A}(\rho)\big)^{-m}\int_{B_{\rho/2}}|D(\eta w^{-1})|^m\\&\leq C\big(M(\rho)+\overline{A}(\rho)\big)^{1-m}\Big(M(\rho)-M\left(\frac{\rho}{2}\right)+\overline{A}(\rho)\Big)^{m-1}\rho^{n-m}.\end{aligned}$$

Thus, we find

$$M\left(\frac{\rho}{2}\right)\leq\frac{C-1}{C}\left(M(\rho)+\overline{A}(\rho)\right)$$

for all $\rho\leq R$ where R depends on r_0 from (2.1), and it follows from Proposition 3.6 that

$$M(\rho)\leq C\left(\left(\frac{\rho}{R}\right)^{\alpha'}M(R)+\overline{A}(\rho^\tau R^{1-\tau})\right)$$

for any $0<\tau<1$ and $\rho\leq R$ with an exponent $\alpha'>0$. Since

$$\overline{A}(\rho^\tau R^{1-\tau})\leq C(\rho^\tau R^{1-\tau})^{\alpha''}$$

with $\alpha''=\min\{\frac{\lambda-\overline{\lambda}}{(m-1)p},\frac{\mu-\overline{\mu}}{mq},1\}$ as it follows from (5.15), we have

$$M(\rho)\leq C\rho^\alpha$$

where $\alpha=\min\{\alpha',\tau\alpha''\}$.

Repeating the above procedure with $-u(x)$ instead of $u(x)$, we get finally

$$\sup_{B_\rho(x_0)}|u|\leq C\rho^\alpha \tag{5.16}$$

for all $x_0\in\partial\Omega$ and all $\rho\in(0,R)$.

With Corollary 2.2 and (5.16) at hand, it is standard matter to get Hölder continuity up to the boundary as claimed in Theorem 2.3. For, we will distinguish between various cases for arbitrary two points $x,y\in\overline{\Omega}$.

Case 1: $\mathrm{dist}\,(x,\partial\Omega)\geq R/2$ *and* $\mathrm{dist}\,(y,\partial\Omega)\geq R/2$. Then

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha}\leq H$$

as it follows from Corollary 2.2.

Case 2: $0<\mathrm{dist}\,(x,\partial\Omega),\ \mathrm{dist}\,(y,\partial\Omega)<R/2$. Let $\delta=\mathrm{dist}\,(y,\partial\Omega)$. We have

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha}\leq\frac{\mathrm{osc}_{B_{|x-y|}(y)}\,u}{|x-y|^\alpha},$$

while

$$\operatorname*{osc}_{B_{|x-y|}(y)} u \le C|x-y|^\alpha \left(\delta^{-\alpha} \operatorname*{osc}_{B_\delta(y)} u + 1\right)$$

as consequence of [17, Theorem 4.1]. This way,

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le C \left(\delta^{-\alpha} \operatorname*{osc}_{B_\delta(y)} u + 1\right)$$

for each $x \in B_\delta(y)$ with a suitable exponent $\alpha \in (0,1)$. Pick now a point $y_0 \in \partial\Omega$ with the property $|y_0 - y| = \operatorname{dist}(y, \partial\Omega)$. Since $B_\delta(y) \subset B_{2\delta}(y_0)$, we have from (5.16) that

$$\operatorname*{osc}_{B_\delta(y)} u \le \operatorname*{osc}_{B_{2\delta}(y_0)} u \le 2 \sup_{B_{2\delta}(y_0)} |u| \le C\delta^\alpha$$

whence

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le H$$

for all $x \in B_\delta(y)$ and all $y \in \Omega$ with $\operatorname{dist}(y, \partial\Omega) < R/2$. Further, if $|x-y| \ge \delta$ and $\operatorname{dist}(x, \partial\Omega) < R/2$, take a point $x_0 \in \partial\Omega$ with the property $|x_0 - x| = \operatorname{dist}(x, \partial\Omega)$. Since

$$|x - x_0| \le |x - y_0| \le |x-y| + |y - y_0| = |x-y| + \delta \le 2|x-y|$$

and $u(x_0) = u(y_0) = 0$, we have

$$\begin{aligned} |u(x)-u(y)| &\le |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\ &\le C\big(|x-x_0|^\alpha + |y-y_0|^\alpha\big) \\ &\le C|x-y|^\alpha, \end{aligned}$$

whence

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le H$$

for all $x, y \in \Omega$ with $\operatorname{dist}(x, \partial\Omega), \operatorname{dist}(y, \partial\Omega) \in (0, R/2)$ and such that $|x-y| \ge \delta$.

Case 3: $\operatorname{dist}(x, \partial\Omega) \ge R/2$ *and* $0 < \operatorname{dist}(y, \partial\Omega) < R/2$. It suffices to take a point z lying on the segment with end x and y and such that $\operatorname{dist}(z, \partial\Omega) = R/2$ to get

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le \frac{|u(x)-u(z)|}{|x-z|^\alpha} + \frac{|u(z)-u(y)|}{|z-y|^\alpha}.$$

Thus, the desired estimate reduces to the cases already considered.

Case 4: $y \in \partial\Omega$. It follows from (5.16) that

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le H$$

for all $x \in \overline{\Omega}$ such that $|x-y| < R$, while

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le 2\|u\|_{L^\infty(\Omega)} R^{-\alpha} \le H$$

if $|x-y| \ge R$, as consequence of (2.6).

It remains to take the smallest of the exponents α in the above considerations to complete the proof of Theorem 2.3. □

5.2. Hölder continuity under natural structure conditions. Theorem 2.3 asserts global Hölder continuity of the weak solutions to the Dirichlet problem (1.1) under the same hypotheses which ensure global boundedness of the solutions. However, it happens very often that one already disposes of an *a priori* bound for $\|u\|_{L^\infty(\Omega)}$ as consequence, for example, of strong monotonicity of the principal part $\mathbf{a}(x,u,Du)$ with respect to Du, or sign condition on $u.b(x,u,Du)$ (see e.g. [19, 20] and the references therein), etc. What is the natural question to arise in this situation is whether the *bounded* weak solutions to (1.1) remain globally Hölder continuous in Ω if the $|\xi|^{m\left(1-\frac{1}{\ell}\right)}$-growth of $b(x,z,\xi)$ in (2.2) is relaxed to $|\xi|^m$.

More precisely, let us weaken the *controlled* growth assumptions (2.2) to the *natural structure* conditions of Ladyzhenskaya and Ural'tseva. In other words, let $\varphi \in L^{p,\lambda}(\Omega)$ with $p > \frac{m}{m-1}$, $\lambda \in (0,n)$ and $(m-1)p + \lambda > n$; $\psi \in L^{q,\mu}(\Omega)$ with $q > \frac{mn}{mn+m-n}$, $\mu \in (0,n)$ and $mq + \mu > n$, and suppose there exist a non-decreasing function $\Lambda(t)$ and a non-increasing function $\gamma(t)$, both positive and continuous, such that

$$\begin{cases} |\mathbf{a}(x,z,\xi)| \leq \Lambda(|u|)\left(\varphi(x) + |\xi|^{m-1}\right), \\ |b(x,z,\xi)| \leq \Lambda(|u|)\left(\psi(x) + |\xi|^m\right) \end{cases} \tag{5.17}$$

and

$$\mathbf{a}(x,z,\xi)\cdot\xi \geq \gamma(|u|)|\xi|^m - \Lambda(|u|)\varphi(x)^{\frac{m}{m-1}} \tag{5.18}$$

for a.a. $x \in \Omega$ and all $(z,\xi) \in \mathbb{R}\times\mathbb{R}^n$.

Indeed, a *bounded weak solution* to (1.1) is a function $u \in L^\infty(\Omega)\cap W_0^{1,m}(\Omega)$ such that

$$\int_\Omega \mathbf{a}(x,u(x),Du(x))\cdot Dv(x)\,dx + \int_\Omega b(x,u(x),Du(x))v(x)\,dx = 0$$

for each test function $v(x) \in L^\infty(\Omega)\cap W_0^{1,m}(\Omega)$.

It is worth noting that in the proof of Corollary 2.2 above, we reduced (2.2) and (2.4) just to (5.17) and (5.18), respectively. Further, it is easy to check that the result of Lemma 5.1 remains valid for *bounded* weak solutions to (1.1) if (5.17) and (5.18) are required instead of (2.2) and (2.4). This way, we have

Theorem 5.2. *Under the hypotheses* (2.1)*,* (5.17) *and* (5.18)*, each* bounded *weak solution of the Dirichlet problem* (1.1) *is Hölder continuous in* $\overline{\Omega}$ *with Hölder exponent and constant depending on the data of* (1.1) *and on* $\|u\|_{L^\infty(\Omega)}$.

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